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Social Networks: Models of Information Influence, Control and Confrontation

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Preface: Phenomenon of Social Networks

Networks had existed since the old days: the road network in the ancient Rome, mail networks in the Middle Ages, or railway, telegraph, and telephone networks. Among recent examples, we should mention telecommunication networks. Each new type of networks had facilitated communication of people and hence promoted progress.

At the same time, as any phenomenon the development of networks has positive and negative features. For instance, some scientists are predicting the future formation of a new "slaveholding society" in which power will be gradually seized by global networks and corporations; moreover, this process is running now. Such structures will control each individual in order to fulfill certain requirements. Even the term of "netocracy" [15] has appeared in literature—a new form of society control in which the basic value is not tangible resources (currency, immovable property, etc.) but *information* as well as different structures to store, process and transmit it. For example, the concepts of corporate brand and logo, corporate style, corporate ethics, corporate parties, corporate holidays (or retreat) have become widespread. Also note shopping in corporate stores. Behind all these concepts there is a drive to keep a man, his family and social environment in full view, under control. Surveillance cameras are being mounted everywhere—streets, banks, supermarkets, etc. Almost every man with his personal data is included in tens of databases and databanks. Much personal data can be found on the Internet, and we may even know nothing about their existence and availability …

Among network resources, a gradually growing role is played by *online social networks*: in addition to the functions of communication, opinions exchange and information acquisition, in recent time they have been intensively used as the objects and means of informational control and an arena of informational confrontation. In fact, they have become a considerable tool of informational influence, particularly for a proper manipulation of an individual, social groups and the whole society, as well as a battlefield of information warfare (cyberwars).

Social Networks. In this book, we consider models of social networks that have become widespread recently as informal communities—a tool for communication, opinions exchange and information acquisition. At qualitative level, *a social* *network* is understood as a social structure consisting of a set of *agents* (subjects, individual or collective, e.g., persons, families, groups, organizations) and also a set of *relations* defined over it (an aggregate of *connections* among agents such as acquaintance, friendship, cooperation, communications). Formally, a social network represents a *graph* $G(N, E)$ in which $N = \{1, 2, \ldots, n\}$ denotes a finite set of nodes (agents) and E a set of edges reflecting the interaction of different agents. Numerous examples of social networks will be given below.

Social networks facilitate, first, the organization of *social communications* of people and, second, the satisfaction of their basic *social needs*. It is possible to identify two intersecting treatments of social network—as a social structure and its specific Internet implementation.

Sociometry, a descriptive framework for social groups in terms of graph theory, was pioneered and further developed by J. Moreno. The concept of social networks was introduced in 1954 by sociologist J. Barnes in [16] and disseminated through scientific community (not only among sociologists[1]) since the early 2000s, following the tremendous advancement of Internet-based technologies. Presently there is a shortfall in a systematic description of network analysis methods and algorithms for modern applied research.

Speaking about the attractiveness of social networks, we may separate out the following **capabilities** for users:

- information acquisition (particularly, revelation of resources) from other social network members;
- verification of different ideas through interactions within a social network;
- the social benefits of contacts (involvement, self-identification, social identification, social adoption, etc.);
- recreation (leisure, pastime).

The keywords of almost any social network model are agent, *opinion*, *influence/trust*, and *reputation*—see Fig. 1. These concepts will be rigorously defined below although everybody knows their common meaning.

Examples and Classification of Opinions for Online Social Network Members. A factor that determines the attractiveness of online social networks for users is the capability to express their opinions (to judge or give an assessment of some issue), see Figs. 2–4. Generally, an opinion is expressed in *text form* as follows.

[1] The structure of social objects had been intensively studied in sociology since the 1950s, simultaneously with an active use of graph theory in sociometry. International Network for Social Network Analysis (INSNA), the professional association for researchers interested in social network analysis, was established in 1977. Social Networks, an international journal of structural analysis, was launched in 1978. We also mention other electronic periodicals such as Connections, Journal of Social Structure, etc. This book does not claim for a complete overview of numerous social network analysis results obtained in sociology (e.g., see [85, 118, 220]).

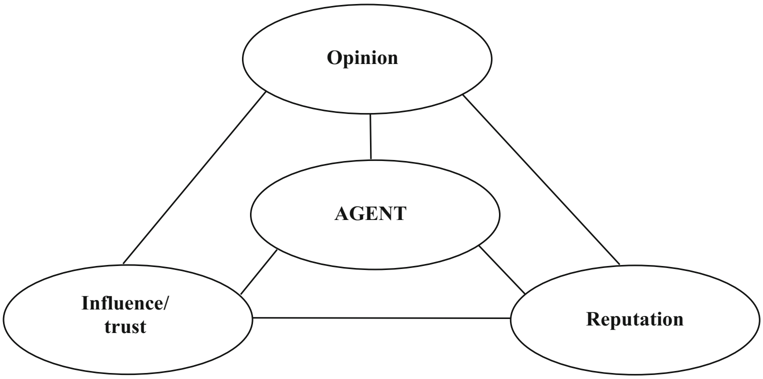

Fig. 1 Basic concepts of social network model

(1) Private correspondence (*text*). User A, "It is cold today. I think it will be about—30°C tomorrow and also this week." User B, "Cold cannot be for so long."
(2) In a blog or forum at the web page of *messages or comments*.

An example is an opinion expressed on a car forum (http://www.drive.ru). "The question is, "How fast does my car go from 0 to 100?" A common answer is, "The devil only knows!" But any owner of Audi RS6 would surely answer, even in a dream, "Four and a half seconds…Well, no! Four and six-tenths seconds." Here the opinion is also a real value.

Alternatively, an opinion can be expressed *using special mechanisms* implemented by social network developers, e.g., as follows.

(3) *Statements about competencies* (see *LinkedIn*).
(4) *Opinion poll*. A user has to choose an alternative from a fixed set, thereby expressing his/her opinion.
(5) Assessment of somebody or something, depending on the thematic scope of a social network. For example, the assessments of movies using *the 10-rating scale* at https://www.imdb.com, see Fig. 4.

A classification of different types of opinions is illustrated in Fig. 5.

This book does not consider mathematical models of the social networks with non-numerical (descriptive text, a fixed set of options) and multicriteria opinions of members. All other cases—see the solid lines in Fig. 5—will be described below.

Properties of Social Networks. For proper modeling of social networks, the mutual influence of their members and opinion dynamics, it is necessary to consider a series of factors (effects) occurring in real social networks. Generally speaking, real social networks may have the following effects and properties caused by the characteristics and demands of agents (who exert influence and are subjected to

Fig. 2 FIFA World Cup post in Facebook: Example of opinion

Fig. 3 Example of opinion poll in Facebook

influence), the character of their interactions, and the internal properties of a given social network[2]:

(1) individual opinions of agents;
(2) variable opinions under an influence of other network members;
(3) different significance of opinions (influence, trust) of given agents for other agents;
(4) different degrees of agent's susceptibility to influence (conformity, stability of opinions);
(5) an indirect influence through a chain of social contacts. Smaller indirect influence for higher "distance" between agents;
(6) opinion leaders (agents with maximal influence), formalization of influence indexes;

[2] The keywords in the list below are underlined.

Fig. 4 Movie assessment in IMDB

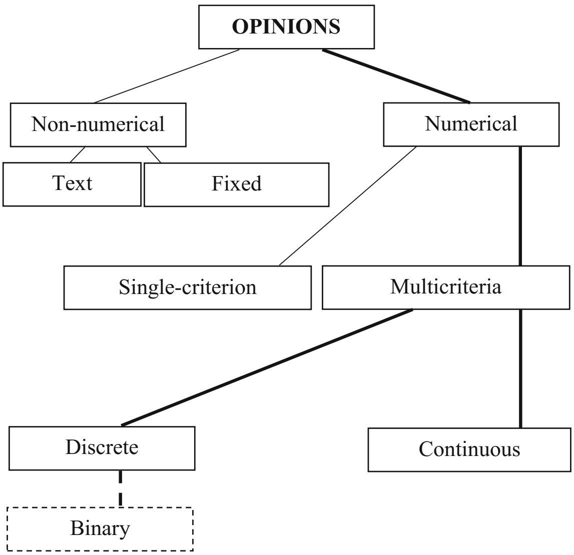

Fig. 5 Classification of different types of opinions

(7) a threshold of sensitivity to opinion variations of a neighborhood;
(8) local groups (by interests, by close opinions);
(9) specific social norms;
(10) social correlation factors (common for groups of agents);
(11) external factors of influence (advertising, marketing events) and external agents (mass media, product suppliers, etc.);
(12) stages—typical phases of opinion dynamics for social network members (e.g., diffusion of innovations);
(13) avalanche-like effects (cascades);
(14) the influence of structural properties of social networks on opinion dynamics:

   - an agent with more connections has wider capabilities to influence the whole network through his/her neighborhood (on the one hand) but higher susceptibility to an external influence (on the other hand);
   - clustering (an agent having active neighbors with dense connections changes his/her state with higher probability; also see the related concept of *strong tie*);
   - local mediation (an agent with higher degree of mediation contributes more to the spread of opinions/information from one part of a social network to another (the role of information broker) but has smaller influence on his neighbors; also see the related concept of *weak tie*[3]);
   - a social network of small diameter has a short chain of opinions spreading;

(15) active agents (with purposeful behavior);
(16) possible groups and coalitions of agents;
(17) incomplete and/or asymmetric awareness of agents, decision-making under uncertainty;
(18) nontrivial mutual awareness (reflexion) of agents;
(19) game-based interaction of agents;
(20) optimization of informational influence;
(21) informational control in social networks.

These empirical effects and properties, to be discussed in detail below, appear in models that claim for an adequate description of real social networks (see Chap. 2 of the book).

Size and Value of Network. As a matter of fact, social networks attract the interest of researchers, particularly due to the fundamentally new properties of agents' behavior in comparison with a set of noninteracting agents. For example, there is ongoing debate on the concept of *value* (*utility*) of a social network.

[3]For example, see the paper [86] by Granovetter and also the book [36] by Burt. In accordance with Granovetter's model, a social network represents an aggregate of strongly tied clusters (groups) that are united into clusters with weak ties.

The value of a social network is the potential availability of agents for "connection" with another agent if necessary [88]. This value has a well-defined numerical characterization. Consider the American market of phones dialing 911 only; actually, the buyers of such phones pay for potential emergency calls (although they might never use this service). In this case, the potential connection to a single agent has a value (the price paid for the phone). Hence, the potential connection to very many agents must have an appreciably higher value.

D. Sarnoff, the founder of *NBC*, was among the pioneers of the value concept of a social network. Sarnoff's law states that the value of a broadcast network is directly proportional to the number of viewers n.

Following the development of local computer networks, a father of Ethernet R. Metcalfe established that the value of a telecommunications network is proportional to the squared number of connected users n^2, see Metcalfe's law [195]. The explanation seems simple: each agent in a network is connected to $(n - 1)$ other agents and the value of the network for this agent is proportional to $(n - 1)$. The network includes n agents totally, and its value is proportional to $n\ (n - 1)$.

The appearance of Internet introduced corrections in social network evaluation. D. Reed [179] acknowledged the above two laws but refined them by an additional term associated with the subgroups of Internet users (Reed's law). This term has the form $2^n - n - 1$, being defined as the number of subsets (subgroups) in the set of n agents except for singletons and the empty set. For each of the laws described, apply a proportionality coefficient a, b and c to obtain the following expression for the value of a social network with very many users n: $a\ n + b\ n^2 + c\ 2^n$.

The late 1990s were remarkable for the mass downfall of dot-com companies. Subsequently, researchers suggested moderate estimates for the real growth of the value of social networks. The work [31] animadverted on Metcalfe's and Reed's laws and estimated the value growth by $n \ln(n)$. A major argument in support of this law (known as Zipf's law) is that, in contrast to the above three laws, the values of connections are ranked. Consider an arbitrary agent in a social network of n members; let this agent have connections with the other $n - 1$ agents of the values from 1 to $1/(n - 1)$. For large n, the contribution of this agent in the value of the whole network makes up $1 + \frac{1}{2} + \cdots + \frac{1}{n-1} \approx \ln(n)$.

Summation over all agents shows that the value of the whole network is proportional to $n \ln(n)$. However, this framework leads to a series of open questions. For instance, why the values of connections have exactly the "uniform" distribution among other agents? And so on.

All the laws mentioned (perhaps, except for Sarnoff's law) are subjected to criticism, and investigators have not still reached a common opinion. Apparently, these debates will span a rather long period of time: it is difficult to formulate a consistent rule explaining the phenomenon in the maximum level of generality without focus on details.

There is another critical remark on the value laws of social net-works. Obviously, the total value of two isolated social networks must coincide with the sum of their values since additional value vanishes in the absence of connections between them. But the laws under consideration do not satisfy such an additive property.

The value of a social network can be described in probabilistic terms with the above additivity. As a characteristic that depends on the potential connections of all agents, the value of a social network must be an increasing function of the number of admissible configurations (potentialities) of these connections in the network. Indeed, the phone market example illustrates that an increase in the number of potentialities (in case of need) raises the value of a network. Denote by $m \in \mathbb{N}$ the number of such admissible configurations and by $f : \mathbb{N} \to \mathbb{R}$ the value of a network, where $\mathbb{N}$ and $\Re$ are the sets of all natural and real numbers, respectively. Then *the monotone property* (nonincreasing values for higher numbers of admissible configurations) can be expressed in the form $f(m_1) \geq f(m_2)$ for all $m_1 \geq m_2$.

Consider two isolated social networks, i.e., any agent from one network has no connection to any agent from the other. Then the value of the union of these networks is the sum of their individual values. The number of admissible configurations in the union of two networks is defined by the product $m_1 m_2$, where m_1 and m_2 indicate the number of admissible configurations in the first and second networks, respectively. So the value of isolated social networks satisfies *the additive property*: $f(m_1\ m_2) = f(m_1) + f(m_2)$.

Imagine that there exists just a single configuration of connections among agents. Then such a social network has zero value, since other potential connections could not be set. Therefore, it is possible to introduce *the normalization property*: $f(1) = 0$.

In probability theory [191], a function that satisfies these three properties is proportional to $\ln(m)$, where m designates the number of configurations, and is called *entropy*. Assume each configuration occurs equiprobably; then there exists a prior uncertainty coinciding with the entropy $\ln(m)$ of the number of configurations. Each concrete configuration eliminates the uncertainty of network connections. Hence, the posterior entropy of each concrete configuration becomes equal to 0. The whole essence of the value of a social network in this interpretation is the degree of prior uncertainty elimination in the network. In other words, agents are potentially available in the sense of the original definition of value.

Consider a network composed of n agents. Renumber all agents in the network. Assume that a network configuration is completely defined by information spreading among agents. For instance, agent 1 receives information from agent 2, agent 2 receives it from agent 3 and so on. Agent n receives information from agent 1. The other configurations are the result of different permutations of agents in the initial configuration. As easily demonstrated, there exist $m = n!$ such network configurations. For a large number of agents n, Stirling's approximation formula [59, 185], leads to the following value of a social network (in the sense of entropy): $\ln(n!) \approx n \ln(n) - n$.

In comparison with Zipf's law, this result corresponds to a more moderate growth of network value proportionally to $n\ \ln(n)$.

As for practical implementations, today there exists a whole class of online social networks united by the same technology *Web* 2.0 [170].

In accordance with O'Reilly's definition, Web 2.0 is the network as platform, spanning all connected devices; Web 2.0 applications are those that make the most of the intrinsic advantages of that platform: delivering software as a continually-updated service that gets better the more people use it, consuming and remixing data from multiple sources, including individual users, while providing their own data and services in a form that allows remixing by others, creating network effects through an "architecture of participation," and going beyond the page metaphor of Web 1.0 to deliver rich user experiences. So, Web 2.0 is remarkable for the principle of users' involvement in the filling and multiple verification of content.

In this definition, like in the above-mentioned laws, a key factor is the interaction of very many agents who increase the value of a social network (modern social networks may cover tens of millions of users). Having this factor in mind, one should employ the well-developed apparatus of statistical physics and information theory to describe the behavior of large-scale systems in probabilistic terms.

Assume the behavior of an agent in a social network depends on several **factors** (see Fig. 6), namely,

- *the individual factor*, i.e., the agent's inclination (preferences) for certain actions;
- *the social factor*, i.e., the agent's interaction with other agents and their mutual influence);
- *the administrative factor*, i.e., *control actions* applied by a Principal to the agent.

The agents subjected to some of the factors are called *dependent* on these factors. If at least the social factor affects the agents, a corresponding network is termed a *nondegenerate social network*. The agents subjected to none of the factors are called *independent*. Finally, if the agents have no dependence on the social factor, a corresponding social network is called *degenerate* [88].

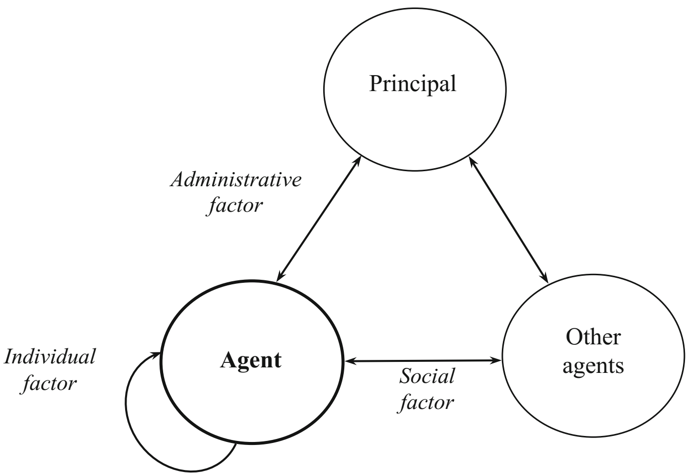

Fig. 6 Factors affecting agent's behavior in social network

It is possible to draw analogies with models in *thermodynamics* and *statistical physics* [185] as follows. A degenerate social network with independent agents matches an ideal gas; a degenerate social network with dependent agents, a polyatomic gas. Next, a nondegenerate social network matches other substances with existing interactions among particles (the mutual influence of agents). Finally, a network with/without control actions matches the presence/absence of disturbances, e.g., from an external field (the influence of a Principal).

There are the following correlations with *information theory* [197]. A degenerate social network corresponds to message coding without penalization while a nondegenerate one to message coding with penalization. Nonadditive penalties describe the mutual interaction between agents; additive penalties, the influence of a Principal. Such models were considered in Chap. 2 of the book [88].

Influence. Control. Confrontation. As mentioned above, online social networks have been intensively used as the objects and means of informational control and an arena of informational confrontation. Therefore, all models that consider *the awareness of agents* (i.e., their information at the moment of decision-making) are traditionally divided into three nested classes—informational influence, informational control, and informational confrontation (see Fig. 7).

Informational influence models study the relationship between the behavior and awareness of an agent (ergo, informational influences). An informational influence model can be used to formulate and solve *informational control problems*: which informational influences (applied by a control subject) will guarantee a required behavior of a controlled subject. Finally, being able to solve informational control problems, we may model *informational confrontation*—the interaction of several subjects with noncoinciding interests that have informational influences on the same controlled subject. As a matter of fact, informational influence models (or *social influence* models in terms of sociology and social psychology) were thoroughly examined for more than 50 years. Meanwhile, the mathematical models of informational control and informational confrontation in social networks (not to mention these problems together, see Fig. 7) were underinvestigated.

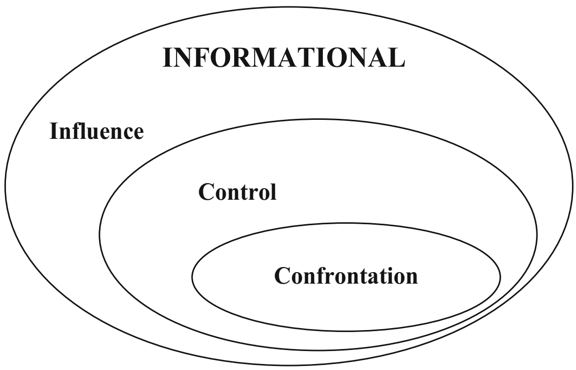

Fig. 7 Informational influence, control, and confrontation

Section 2.1 considers the informational influence of agents on their opinions in social networks (generally speaking, the model below involves the traditional framework of Markov chains for social network analysis, see [66, 98] and also [180]). The structure of a social network is described using the concepts of *community* (a set of agents that undergo no influence from other agents outside it), *group* (a community of agents in which each agent has influence on or undergoes influence from each other agent of this community, directly or indirectly), and *satellite* (an agent that has no influence on any group). Assume at least one agent in each group more or less trusts in his/her opinion. Then it turns out that, in the final analysis, the opinions of all satellites are defined by the opinion of groups; moreover, the opinions of agents within groups converge to the same value. (General necessary and/or sufficient conditions of convergence—the regularity of Markov chains, etc.—can be found in [70, 76, 112] while the communication structure of agents, including its role in convergence, was considered in [3].) For such social networks, the following statement of *informational control problems* seems quite natural: for a small set of key agents in a given network, find variations of their opinions, reputation and/or trust so that the resulting opinion dynamics yield required opinions for all members of the network or its part. Some informational control models are described in Sects. 2.2–2.6.

Recall that we have identified three components of social network models—opinion, trust, and reputation (see Fig. 1). *Control* is a purposeful influence on a controlled system in order to guarantee a required behavior [165]. So the object of control in a social network can be the opinions of agents, their reputation and mutual trust. Informational control models for agents' opinions are considered in Sects. 2.2 and 2.3; informational control models for agents' reputation, in Sect. 2.4; informational control models for agents' trust, in Sects. 2.4 and 2.5; an informational control model with the incomplete awareness of a Principal, in Sect. 2.6. Next, Sect. 2.7 introduces a model of actions spreading through a social network and also an associated influence calculation method. This model estimates the influence and influence levels of different agents based on existing observations of real social networks, which can be further used in opinion formation models.

In addition, we state and analyze the associated game-theoretic problem of *informational confrontation* among several players in a social network (see Sects. 3.1 and 3.2). Two cases can be separated out as follows. If several players choose their informational influences simultaneously, then their interaction is well described by a *normal form game* (see Sects. 2.2 and 3.1). The case with a fixed sequence of moves leads to *an attack-defense game*, which is considered in Sect. 3.2.

Section 3.3 presents another approach to the game-theoretic modeling of informational confrontation that is superstructured over threshold models of mob. Such models describe a mob as a set of agents with the so-called conformity behavior: their binary choice (to be active or passive) depends on the decisions of other agents. In this context, we also refer to a survey of threshold models in Chap. 1.

Safety. Besides the discussed capabilities, like any large social phenomenon online social networks cause a series of problems for users: diversion from the reality; a lack of live communication; much time spent on communication, in particular with unfamiliar people (which might affect study, work, and personal life); and so on. Still these problems have not reached an adequate level of formalization, and they will be omitted in the book.

As emphasized earlier, social networks can be a tool of informational control (manipulation, implicit control), which inevitably leads to a dual problem—the analysis and control of *informational safety* of social networks.

Another well-known fact is that information systems have become an integral support and implementation tool for managerial decisions at all levels—from operators of industrial processes to country leaders. Therefore, informational safety might (and should) be supplemented by *the social safety of information and communication technologies* (ICT), i.e., the safety of ICT users, their groups and the whole society from informational influences and the negative consequences of managerial decisions based on modern ICT.

Consequently, it is important to examine the issues of informational influence, informational control and informational confrontation, in particular in the following aspects:

- informational influence on separate persons, social and other groups, and the whole society;
- purposeful influence (informational control), including such influence through mass media;
- struggle for high informational influence levels and formation of required opinions in a society;
- the safety of managerial decisions depending on available information;
- informational confrontation (including implicit confrontation) at international, national, regional, municipal, sectoral, and corporate levels.

Some safety control models for social networks will be considered in this book. However, this field has not attracted proper attention of researchers, and the design and analysis of **social safety models for ICT** (including safety models of social networks) seems to be a topical branch of future investigations.

Structure of This Book. The Introduction considers in brief game-theoretic models as a formal description for interacting elements of network structures. Chapter 1 gives an analytic survey of informational influence models for social networks. Next, Chap. 2 presents original results of the authors and their colleagues on the design and analysis of theoretical models of informational influence and control in social networks. Chapter 3 is dedicated to original theoretical models of informational confrontation in social networks. At the end of each section in Chaps. 2 and 3, we outline some promising fields of further research, which explains the absence of conclusions in this book.

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Introduction: Games and Networks

Throughout many years, game- and graph-theoretic models have been successfully used to describe complex systems. This book is dedicated to models of social networks formalized by graphs while the problems of informational control and informational confrontation are stated, particularly, in terms of game theory. For a proper characterization of this class of models, consider in brief the modern correlations of game- and graph-theoretic models.

In accordance with [153], game theory studies mathematical models of conflict and cooperation among rational subjects (players) that make decisions. The results obtained in game theory have found numerous applications in different fields—sociology [180, 194], economics [144, 151, 153], organizational control [80, 165], ecology [35, 180], military science [207], etc.

As a theoretical discipline, *graph theory* is a branch of applied mathematics that analyzes the properties of finite sets with given relations among their elements. In the sense of applications, graph theory allows describing and exploring many technical, economic, biological and social systems; a series of examples were discussed in [112, 180, 207].

Graphs and Games. As a matter of fact, game theory and graph theory possess a strong correlation. Some examples where the framework and results of graph theory are adopted in game-theoretic setups include the following:

– a tree graph defines the structure of decision-making in an extensive-form game [153];
– a graph (with nodes as players) determines the structure of admissible coalitions [80];
– a graph models a "search game" in discrete time (here nodes are the positions of players and edges correspond to admissible transitions) [69];
– a directed graph describes how the payoff functions of certain agents depend on the actions of other agents (e.g., Nash equilibrium can be achieved on connected graphs); in the general case, a graph reflects the awareness structure of players [168] or the structure of communications among players [162];

– a graph characterizes permanent or temporal relations (informational or technological relations, subordinacy relations, etc.) among players [79, 162];

… and so on.

Moreover, we should mention theory of network games, a relatively young branch of game theory dating back to the late 1970s. It focuses on the formation of network structures—stable relations among players—under their noncoinciding interests and/or different awareness (see the survey [77] and the monograph [112]).

In this context, two terminological remarks are needed as follows. First, in network games the term "network" has a wider meaning than in graph theory, as almost any graph is called a network. Second, along with the term "network games," a growing number of authors operate the term "network formation games." Actually, it better fits the whole essence of a game that produces a network connecting players. This trend allows for a simple explanation: network games can be treated as network formation games (see Fig. 8) and as network-based games with fixed-structure networks. Next, the network-based games comprise the following classes (see Fig. 8):

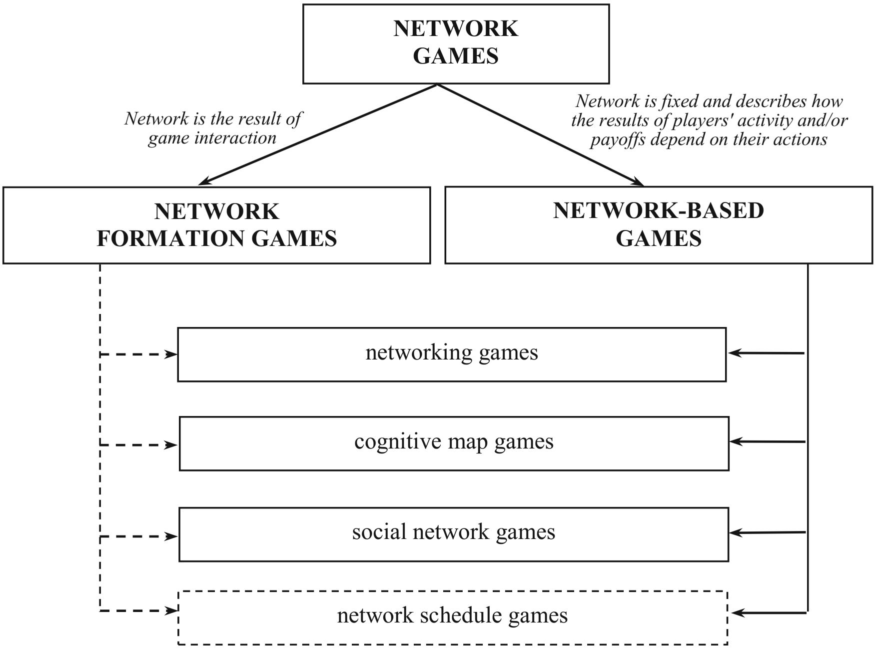


Fig. 8 Network games

- *networking games*[4];
- *cognitive map games*;
- *social network games*;
- *network schedule games.*[5]

At qualitative level, the distinction between network formation games and network-based games is that, in the former, players choose variables connected with paired interaction among them; in the latter, players choose variables describing network nodes (e.g., factor values in cognitive map games, agents' opinions in social network games, etc.). Perhaps, a formal unification of these models will be reasonable for future investigations, see dashed line in Fig. 8. A potential benefit from such an initiative is that many network formation games (e.g., information communication models in multiagent systems) require a model of network dynamics for payoff evaluation, similarly to network-based games. The unification of these models would lead to a two-step game in which players form a network (Step 1) and use this network to transmit information, resources, etc. (Step 2) in accordance with the concept of network-based games.

Network-Based Games. Recent years have been remarkable for the appearance of different practical setups for the description and analysis of an agents' interaction in which the result of interaction (or the relationship between chosen actions/strategies and payoffs) is defined by a certain network (graph-theoretic) model. As emphasized above, such games are called network-based games. Consider a series of examples.

Cognitive map games [158] involve cognitive maps [9], i.e., weighted directed graphs in which nodes are factors (evaluated via a continuous scale or a fuzzy scale) and weighted or functional arcs reflect the mutual influence of factors. Such games are used for taking into account the causal relations and mutual influence of factors as well as for the dynamic modeling of weakly formalizable systems [126]. Cognitive models have various applications, see [9, 180]. For an introduction to this field of research, we recommend the classical monographs [9, 180].

The main objective of using cognitive maps consists in qualitative analysis, mostly relying on simulation modeling of situation dynamics (trends, directions of factors variation, scenarios, etc.). In other cases, qualitative analysis is used for solving inverse control problems in analytic form. For example, by describing the correlation of factors as a system of second-order linear differential equations and specifying some initial conditions, one can analyze the dynamics of factors, steady-state values and so on. Therefore, it is possible to consider all these aspects

[4] This class of games mostly has transportation and telecommunications interpretations (see the monograph [184], the pioneering paper [209] and also the survey [64]). Here network is a tool and/restriction for players' interaction.

[5] Unfortunately, still researchers have not paid due regard to this class of games. It can be characterized as games of subjects allocating certain resources to implement operations within the network schedule of a project. In other words, network schedule games are a game-theoretic generalization of resource allocation problems defined over networks; these problems represent classical examples in network planning, scheduling and control.

from the viewpoint of persons interested in certain situation development or to explore the noncoincidence of goals pursued by different subjects. A correlation model of factors being available, a game-theoretic setup can be considered as follows. Assume players can influence the initial values of factors (e.g., for each player there is a given set of controlled factors) and their payoffs depend on the steady-state values of factors. An example of such a linear game was examined in [158].

In *social network games*, nodes are agents (i.e., the members of a social network) and weighted arcs reflect their degrees (levels) of mutual trust (influence), see the monograph [112] and Chap. 2 of this book. The opinion of each agent is formed under the influence of his/her initial opinion and the opinions of other agents depending on their mutual trust (the opinions dynamics obey a system of linear differential or difference equations). In addition to agents, the model includes players which can influence the former, including their interaction. This means that the players are able to *control* agents. With a known relationship between the initial opinions and the structure of a social network (on the one part) and the final opinions (on the other part), it is possible to rigorously formulate and solve the following problem. Which initial opinions of agents and relations among them (including the structure and the degrees of trust) should be formed by the players for reaching an equilibrium of the game in some sense? Note that this book is dedicated to social network games in accordance with the above-mentioned correlation of the game- and graph-theoretic models.

Among other examples, we mention the usage of Petri nets [218].

All examples above (generally speaking, all network-based games) have a common feature as follows. The connection between the players' actions and the result that defines their payoffs is described by a dynamic system, a system of differential equations, etc. within a rather simple network. To put it bluntly, a network represents the model of players' interaction (factors' interaction, etc.). The next step is to analyze the properties of a corresponding dynamic system, which leads to some classical game-theoretic setup (in the general case, a dynamic game [158, 216]). Interestingly, networking games stand aside: they have almost no dynamics and the solution is Wardrop equilibrium [209].

Furthermore, consider a network as a control object. Studying the properties of this network, being able to describe its dynamics depending on certain parameters and extracting controlled variables (the parameters varied by a Principal), we can pose and solve different control problems. This point should be elucidated.

Control Problem. This paragraph presents a qualitative general statement for the control problem in a certain system. Consider a control subject (*Principal*) and a controlled system (*controlled object*). The state of the controlled system depends on external disturbances, the control actions of the Principal and possibly on the actions of the controlled system itself (this is the case for active controlled objects—subjects—arising in socioeconomic or organizational systems, see Fig. 9). The Principal's problem is to perform control actions (solid line in Fig. 9) for ensuring a required state of the controlled system under existing information about external disturbances (dashed line in Fig. 9).

Fig. 9 Structure of control system

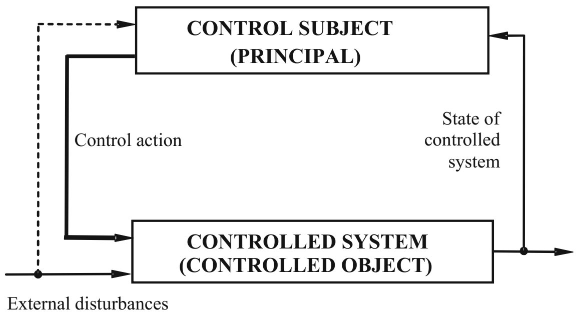
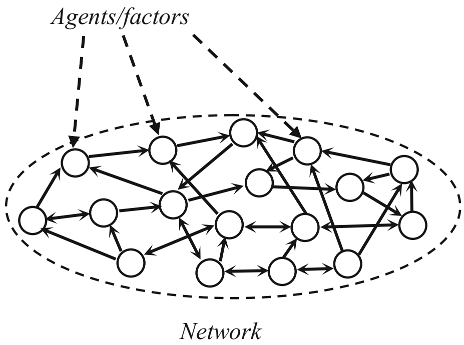


Fig. 10 Network as model of controlled object



A controlled system can be described in different ways (e.g., a system of differential equations, a set of logical rules, etc.), which reflect the dependence of its states on external factors, control actions, past states, and so on. In particular, a formal description may involve a network model; for example, Fig. 10 shows a network in which nodes correspond to the components of the state vector (agents, system participants) while arcs characterize their mutual influence.

The book [165] suggested a classification of control problems depending on the object affected by control actions, which identified the following types of control:

– staff control (control of elements belonging to the staff of a controlled system);
– structure control (control of connections among system elements);
– institutional control (control of the constraints and norms of activity for system elements);
– motivational control (control of preferences and goals);
– informational control (control of the awareness of system elements, i.e., control of all information available to them at the moment of decision-making).

In its network interpretation where a controlled object is a graph (composed of passive nodes without individual preferences and awareness), control actions are purposeful influences on the following components of a controlled object (see Fig. 11):

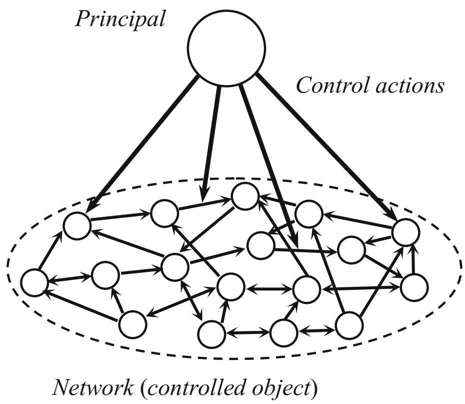


Fig. 11 Control of object described by network

- the staff of a controlled system (i.e., control consists in elimination or addition of nodes);
- the structure of a controlled system (i.e., control consists in elimination or addition of arcs);
- the values of parameters corresponding to graph nodes (the values of states) and its arcs (the values of parameters reflecting the connections among system elements).

As for the social networks considered in this book, most of the modern control models describe an influence on graph parameters, almost not affecting the staff and structure of a given network. Therefore, control problems for the staff and structure of social networks are an interesting field of future investigations, both in terms of their statement and solution methods.

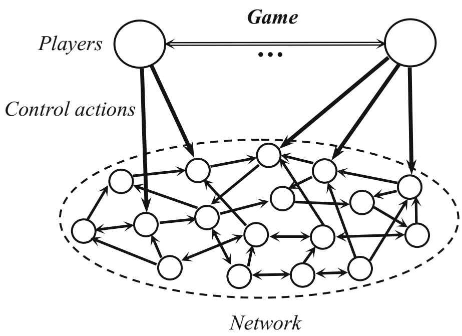
Note that control of a network is an independent nontrivial problem which may employ the framework of operations research and optimal control theory. Furthermore, a separate issue concerns stability, either the Lyapunov stability of a controlled system, or the stability of solutions with respect to model parameters (the well-posedness of a control problem, etc.) [200].

Now, complicate the model by assuming that there are, at least, two Principals. Each of these *players* can apply control actions to some components of a controlled object, as illustrated in Fig. 12.

If the preferences of players (their "efficiency criteria" or goal functions) depend on the state of a controlled object (in the general case, the latter is defined by the actions of all players), we naturally obtain a **network-based game** defined above.

Assume the set of players, the sets of their admissible strategies, goal functions (defined over the set of actions and network states) and the network itself (with all its properties, including the relationship between players' actions and network states), the awareness of players and their decision-making form common

Fig. 12 Network-based game (confrontation)



knowledge among all players.[6] A set of the above parameters determines a *dynamic game* (see the surveys in [207, 216, 153]). This means that, in the case under consideration, a network-based game can be reduced to a dynamic game.

An exploration of network-based games includes the following stages:

(1) the description of a given network, including dynamics analysis;
(2) the description of the set of players, their preferences, awareness structures, the sets of admissible strategies and controlled parameters;
(3) the reduction of a given network-based game to some game-theoretic model (extensive- or normal-form game, cooperative game, etc.).

This stage exhausts the network specifics, passing on the baton to classical game-theoretic analysis. Of course, the results of such an analysis should be given some interpretation in network terms. In other words, the problem is to transform an initial network-based game into an appropriate game for using a rich arsenal of game-theoretic methods.

The whole variety of network models and games defined over them needs a suitable classification system. It is possible to suggest two almost independent classification systems—from the viewpoint of games and also from the viewpoint of networks over which these games are defined.

Classification of Network-Based Games. First, we will classify network-based games from the viewpoint of game theory, specifying classification bases and admissible values of classification attributes.[7]

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[6]In other words, the listed parameters are known to each player, each player is aware of this fact, each player knows that the other players are aware of this fact, and so on—generally speaking, the process of such reasoning is infinite [168]. If this assumption is rejected, reflexive network-based games have to be considered.

[7]For each classification basis, it is possible to distinguish a greater number of subclasses (the number of values of classification attributes). An alternative approach widens the circle of classification bases by importing them from optimal control theory, operations research, etc.

1. The type of a dynamic system (for network models with dynamics). For this classification basis, we may discriminate between *linear games* (in which the variations of node values depend linearly on the values of other nodes, their variations and control actions) and *nonlinear games*.
2. The awareness of players. Here the admissible values of classification attributes are as follows: (a) the parameters and current results of a game form common knowledge; (b) common knowledge is absent. In the latter case, we have *reflexive network-based games* (see the description of reflexive normal-form games in [168]). This class of games can be an efficient tool to model informational confrontation, information warfare, etc. [126, 168]. *The asymmetric awareness of players* may take place depending on the parameters observed by them.
3. The presence or absence of uncertainty (symmetric uncertainty or asymmetric uncertainty when players possess different prior information and this fact forms common knowledge). The deterministic case seems simpler; at the same time, e.g., *network-based games (symmetric) uncertainty* well describe situations of decision-making and/or scenario modeling in uncertain conditions.
4. Discrete or continuous time. If node values depend only on the actions of corresponding players, we obtain classical *differential games*. They represent an intensively developed and fruitful branch of modern game theory, see [216] and also bibliography therein.
5. The structure of players' goal functions. The goal function of each player possibly depends on the dynamics demonstrated by the values of all nodes (trajectories) and his/her actions. In the general case, the payoff of each player is explicitly defined by the actions of all players. There may exist *integral criteria* that describe the player's payoff using a time integral of some function of the game trajectory and players' actions (e.g., with normalization by game duration—*average criteria*). In *terminal criteria*, the payoffs of different players depend on the node values at terminal times. Note a specific set of terminal nodes (goals) can be defined for each player, and so on.
6. Time horizon, for analyzing dynamics and solving associated control problems. We will differentiate between *finite* and *infinite* horizons.
7. The structure of constraints. Certain constraints can be imposed on the individual actions of players only. We also mention *joint activity constraints* [162, 165], and/or individual constraints in constructive form (e.g., bounded time integrals of given functions of players' actions).
8. The foresight of players. In the conditions of complete awareness and common knowledge on a finite horizon, players may simultaneously choose their actions for all future times (the so-called *programmed decision-making*). *The foresight of players*, i.e., the number of future times considered by them, can be smaller than the time horizon. In this case, *sliding decision-making* is required in which players may undertake certain commitments on the choice of their actions.
9. The times of action choice for players. Several situations are possible as follows. First, the so-called *impulse control* in which the actions of players explicitly affect the values of nodes at a single time (e.g., the initial time) or

several starting times, with subsequent relaxation dynamics. Control can be *continuous*, i.e., the actions of players explicitly affect the values of nodes at each time. Finally, control can be *periodic*.[8]

10. The sets of nodes controlled by different players. In the general case, the value of each node in a dynamic game evolves depending on the actions of all players. In special cases, there may exist specific sets of controlled nodes for certain players. The sets of controlled nodes can be overlapping or not.
11. The sequence of moves. Players can make decisions *simultaneously*. The sequence of moves (choice of actions) may differ during a single period, which leads to *multi-step hierarchical games* [71] (two players) or *multi-step multi-level hierarchical games* (three or more players). Alternatively, different players can choose their actions at different times—an analog of extensive-form games or *positional games*.
12. Coalition formation. Making their decisions, players may exchange information, negotiate joint actions and redistribute their payoffs. This leads to a *cooperative game*.

The second system of classification bases for network structures can be described **from the viewpoint of graph theory**. More specifically, network structures may involve [158]:

- *functional graphs* (in which the influence of one node on another is a given function of their values);
- *delayed graphs* (in which any variations in the value of one node cause variations in the value of another with some delay);
- *modulated graphs* (in which the influence of one node on another depends on the value of a third node called a modulated node);
- *hierarchical graphs*;
- *probabilistic graphs* (in which each arc describes an influence realized with a given probability);
- *fuzzy graphs*, etc. Different interpretations of nodes, arcs and their weights as well as different functions determining a mutual influence of nodes generate a variety of network models.

Intermediate Conclusions. Combining different values of attributes for each classification basis above and choosing a certain type of network structure, we can define several types of network-based games and also assign an appropriate class for a specific network-based game.[9]

The existing classification system allows us to generate adjacent problems using the results of investigations for a certain network-based game as well as to extend and/or generalize well-known results to them.

[8] Of course, each player may have a specific sequence of decision-making times at which his/her actions explicitly affect the values of certain nodes.

[9] We recommend the reader to get back to this classification after perusal of the book, which would outline the place of all models considered below.

The current advances in the field of network-based games—a correct reduction of some network-based games to classical normal-form games [94, 158] or reflexive games [168]—seem to be very modest. From a theoretical perspective, further research should be focused on the study and applications of the models of network-based games that have been classified above, namely, nonlinear, reflexive, hierarchical and cooperative models of qualitative decision-making (defined over fuzzy and/or probabilistic and/or functional graphs) in uncertain conditions, etc.

Chapter 1
Models of Influence in Social Networks

In this chapter presents an analytic survey of modern models of social networks as well as establishes a correspondence between different classes of models and the properties of social networks reflected by them (see the Preface). Further exposition is organized as follows.

Section 1.1 (Influence and influence level) includes four subsections. The first subsection considers the existing definitions and models of influence in social networks and also the formal approaches to determine the influence levels of agents. Next, the second subsection deals with the models of diffusion of innovations. The third subsection is dedicated to opinion formation models for social networks. Finally, the fourth subsection discusses models of influence and information spreading.

Section 1.2 (Common knowledge. Collective actions) has the following structure. The first subsection studies the role of awareness. The second subsection is focused on public goods and specialization. The third subsection describes communication and coordination in social networks. The fourth subsection considers social control and collective actions as well as network stability.

Section 1.3 (Models and properties of social networks) summarizes this analytic survey and establishes a correspondence between the properties of social networks reflected by certain models.

1.1 Influence and Influence Level

1.1.1 Influence. Classification of Models

Influence is the process during which a certain subject (the subject of influence) changes the behavior of another subject (an individual or collective object of influence), his/her attitude, intentions, beliefs and assessments (and also the

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resulting actions) using interaction with him/her, as well as the result of this process [73]. *Influence* is the capacity to have an effect on the character, development, or behavior of someone or something, or the effect itself [174]. There exist purposeful and purposeless influence [73]. A purposeful (goal-oriented) influence is an influence that involves such impact mechanisms as persuasion and suggestion. In addition, the subject of influence seeks for definite results (goals, e.g., specific actions) from the object of influence. A purposeless (goal-free) influence is an influence in which an individual does not need definite results from the object of influence (in some cases, the former does not even suspect the existence of the latter). In this book, we will discuss influence together with the concept of influence level, which is the total influence on a given community (network).

As indicated by psychological studies [54], agents in a social network often have insufficient information for decision-making or are unable to process it independently. Therefore, their decisions can be based on the observed decisions or beliefs of other agents (*social influence*). Social influence is realized over two processes—*communication* (contacts, the exchange of experience and information, discussion of certain issues with authority neighbors leads a given agent to certain beliefs, attitude, and opinions) and *comparison* (in pursuit of social identity and social approval, an agent accepts beliefs and actions expected from him/her by other agents in a given situation). During the second process mentioned, an agent wonders, "What would be done by another agent (role model) in my situation?" So, comparing him/herself with this agent, the former assesses his/her own adequacy and plays a corresponding role. Comparison can be also explained by the aspiration for strategic advantage: contrasting him/herself with other agents that have the same positions in a social system, a given agent may introduce or accept innovations for becoming a more attractive object of relations. Note that the communicative approach to influence may lead the agents to the same beliefs but not necessarily to the same behavior. In the case of comparison, however, an agent indirectly copies the actions of other agents. Clearly, the behavior of a given agent is determined by his/her beliefs but also by the limitations he/she is facing. So agents with similar beliefs may behave differently; conversely, agents with different beliefs may act in the same way. In social networks, a visual relationship between the actions of neighbor agents can be defined by social pressure (an action or opinion of a given agent may stimulate other agents to follow him/her) and to a greater extent by other factors of *social correlation*—external environment (same place of residence, same occupation, etc.) or the similarity of agents (e.g., close preferences or tastes). Nevertheless, it is possible to detect an existing influence in a network due to its causal character. In particular, the paper [143] considered tests to identify the social influence factor.

A social network plays crucial role in the spread of information, ideas, and influence among its members. A major issue of informational processes analysis in social networks is to evaluate the influence of different users (some problems associated with networks were discussed in [90]). In fact, there exist several approaches to define the influence and influence level of users as follows. *The structural approach* to modeling and influence evaluation operates the concept of structural centrality from the classical theory of social network analysis (SNA) [6, 62, 198, 210]. Since the 1950s, scientists have been developing different structural centrality indexes (node closeness, node betweenness, edge betweenness and others, see [65]) to describe influence. However, informational interaction within a social network is not always caused by its structure (e.g., see [89]), which forms a considerable drawback of this approach. Also note numerous studies of influence indexes (the Banzhaf power index, the Hoede–Bakker index, etc. [5, 63, 84, 104, 186]) in decision making. *The dynamic modeling approach* relies on a certain model of different informational processes in social networks. As assumed here, influence predetermines the dynamics of informational processes (opinions formation, information spreading, etc.). In this context, we mention Markovian models, threshold models, independent cascade models, Ising models, cellular automata models, epidemic models and others. This approach is used for solving different optimization problems, particularly, the problem of most influential users (in the general case, the problem of a finite set of most influential users) indirectly causing a maximum spread of given information through a social network [124]. *The approach based on actions and interests* (*the actional model*, see Sect. 2.7 of the book) proceeds from the actions performed by social network users (writing posts, comments, etc.) and formalized interests of a control subject (Principal). Finally, *the computational approach* evaluates influence using modified page ranking algorithms and scientometric methods [4] or machine learning methods [178].

In the sequel, we will study dynamic models of informational processes over a fixed network with local interaction rules of its members (interpersonal influence). Less attention will be paid to dynamic models of informational processes with macrolevel variables (e.g., see the macromodels of epidemics in the survey [155] or social communication models in [146, 177]).

Classification of influence models for social networks. We have thoroughly analyzed the literature to identify the following general classes of models.

Optimization and simulation models, which include several subclasses considered in Sect. 1.1, namely,

1.1. the models of diffusion of innovations;
1.2. the models of opinions formation;
1.3. the models of influence and information spreading.

Models 1.1–1.3 (described further in Sect. 1.1) mostly deal with the interaction rules of agents, covering merely a few aspects of the influence network and its properties as well as the relationship between its structure and interaction processes.

Game-theoretic models, which stress main emphasis on the awareness and interconnections of players (agents). The payoff of a given agent (player) depends on the actions of his/her opponents (other players). Each agent seeks to maximize his/her payoff. A series of game-theoretic models will be considered below (in Sect. 1.2 and Chap. 3 of this book), including:

2.1. the models of mutual awareness;
2.2. the models of coordinated collective actions (and public goods);
2.3. the models of communication processes and the problems of minimal sufficient network;
2.4. the models of network stability;
2.5. the models of informational influence and control;
2.6. the models of informational confrontation.

Now, we will describe these classes of models in detail.

1.1.2 Influence and Diffusion of Innovations

In the literature dedicated to social networks, influence has close connection to the concept of *diffusion of innovations* [182]. So we will discuss in brief the corresponding models of diffusion of innovations.

The properties of large-scale systems were considered in [156, 211]. The dynamics of innovations spreading (the share of a population that accepts innovations) is traditionally modeled by the *S*-shaped curve, also known as the logistic function. Actually, this is a characteristic of any infectious process [212], learning process [166], diffusion of innovations [182], see Fig. 1.1. The *S*-curve has the following stages [72]: *innovators* (who accept and use an innovation "in the forefront"), *early adopters* (who perceive an innovation and start using it soon after appearance), *early majority* (who perceive an innovation after innovators and early adopters before the majority of other agents), *late majority* (who perceive an innovation after widespread use), and *late adopters* (who perceive an innovation after all the others). These groups are illustrated in Fig. 1.2 in form of the so-called Bell curve—the derivative of the *S*-curve.

Like many other processes in nature and society, the diffusion of innovations has certain limits, in the first place due to finite resources (the bounded capabilities and capacity of a social system). The *S*-curve contains three phase of development—base formation (slow growth), abrupt growth, and saturation (slow growth). A key

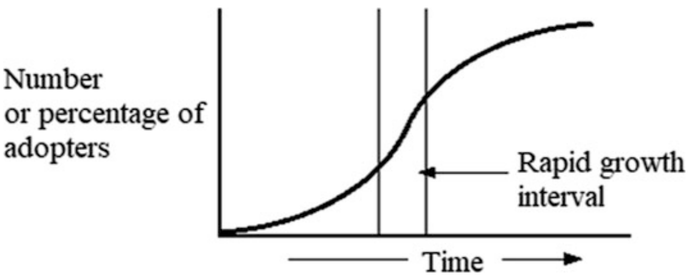


Fig. 1.1 S-curve (logistic function)

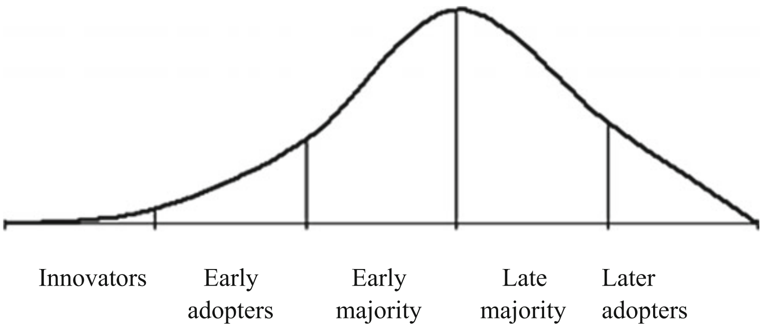


Fig. 1.2 Bell curve

factor that determines the speed of diffusion processes is the interpersonal communication among the adopters of a given innovation and those who is hesitating or knows nothing about it.

Whereas innovators can be characterized as non-conformists (or even originals) and early adopters as the agents susceptible to the social normative and informational influence (or having good sense for promising things), late adopters are difficult to affect and stable agents of a social network. The adoption stages of innovations for different agents were considered in [182].

Very often small variations in the states of network nodes may cause *cascade* (avalanche-like) *changes* (*local*—covering the initiator's neighborhood—and *global*—over the whole network). The *Word-of-Mouth* effects were empirically studied in [74, 135], yet without a detailed structural analysis of networks. The papers [121, 133] examined the interconnection between network structure and group coordination problems but for artificially generated networks. So it is not clear how these results apply to real social networks.

Diffusion of innovations: models of public opinion formation. In such models, public opinion is an innovation. As a matter of fact, this class includes many models. For example, there exist models considering agents as separated objects of influence by mass media [21]. Here the two-step model [120] is widely used, in which mass media first form the opinions of the so-called leaders (the agents with good awareness, high respect or many links) and then leaders do the same for other (common) agents. But the grounds of such a heuristically clear approach seem doubtful. Are the leaders really affecting the whole community through their closest neighborhood? Is their influence crucial? This model also neglects reciprocity: common agents also affect the leaders, and influence can be transferred in more than two steps. Moreover, many mathematical models (e.g., see [10, 55, 56, 87, 193, 217]) do not explicitly assume the presence of opinion leaders or special individuals to obtain the *S*-curve of diffusion of innovations.

Diffusion of innovations: role of leaders. The paper [212] identified the role of leaders in the diffusion of innovations within a simple model of social influence (particularly, how the dynamics of their opinions cause large cascade changes of opinions in a social network). As established there, in most cases leaders have a moderately higher significance than common agents (with some exceptions). Actually, large cascades are generated by the mutual impact of different easily influenced groups of agents.

We will explain this conclusion below. Consider the linear threshold model [212] in which agent i makes binary decision on some issue. The probability that agent i prefers option CB to option CA is increasing with the number of other agents who choose CB. (This fact is well-known in social psychology, although the threshold model does not cover several factors, e.g., reactive resistance [152]). The threshold rule has the form

$$P[\text{choose } CB] = \begin{cases} 1 \text{ if } r_i \geq \phi_i, \\ 0 \text{ if } r_i < \phi_i; \end{cases}$$

where ϕ_i denotes a threshold; r_i is the share of agents choosing CB. Note that the model can be extended using another probability with higher sensitivity to variations of the share r_i.

In addition to the threshold rule that determines the influence of other agents on the decisions of a given agent, the model should include information about the influence network (mutual influence of different agents). As assumed in [212], agent i within a population of size n influences n_i other randomly chosen agents. The number n_i is taken from the influence distribution $p(n)$ with the mean $n_{\text{avg}} \ll n$, which characterizes the influence of agent i on n_i other agents subject to the given issue. In this influence network, all agents may affect each other (directly or indirectly). The authors [212] defined *opinion leaders* as the agents located at the upper decile of the opinion distribution $p(n)$.

Opinion dynamics were also considered in [212] as follows. At the initial step, the agents are passive (in state 0) except for a single randomly chosen *active initiator i* (opinion leader) who is in state 1. This initiator can activate the neighbors, thereby generating a *cascade* in a chain. Assume a large number of *early adopters* (the agents directly connected with the initiator in the network) are also connected with each other. Then a global cascade may occur, although in general such adopters may constitute a small part of the population. A series of experiments was performed in [212] for comparing the average size of a cascade initiated by the opinion leader with the average size of a cascade initiated by a common agent.

Note that the average threshold ϕ equally affects the cascade initiation capability of the opinion leader and a common agent. So a relative comparison of their significance is independent of ϕ. The size of cascades generated by single initiators strongly depends on the average density n_{avg} of the network in the following way. If this value is small, then many agents are susceptible but the network has insufficient density for spreading and merely a small part of the network is activated in final analysis. For large n_{avg}, the network becomes highly connected but for activation the agents need a large number of activated neighbors; in other words, a few initiators cannot generate a global cascade. In fact, global cascades may occur on the medium interval—the so-called *cascade window*. Within this interval, leaders and common agents may initiate cascades. Consequently, the cascade initiation capability of an agent depends on the global structure of the network rather than on his/her individual degree of influence. If cascades are possible in principle for a given network, then any agent can initiate them; otherwise, nobody. This result holds regardless of the value ϕ, which merely "shifts" the cascade window for the leaders and common agents.

As indicated by experiments [212], leaders initiate cascades that are slightly larger than the ones initiated by common agents (their sizes almost coincide), except for narrow limits of the cascade within which the former have considerably higher significance than the latter. On the other hand, leaders may play key role in initiating global cascades as the critical mass of early adopters. If the network has low density (n_{avg} is near the left limit of the cascade window), then early adopters are more influential in average (i.e., $n_i > n_{\text{avg}}$). For the network with high density (n_{avg} near the right limit of the cascade window), we observe the opposite picture—early adopters are less influential in average ($n_i < n_{\text{avg}}$). This phenomenon can be explained in the following way. The agents with high influence (a large value n_i) are less susceptible but, after passing to state 1, they can activate more other agents. However, in accordance with experimental evidence, early adopters are not opinion leaders despite their higher influence against average agents. In other words, early adopters are not always influential enough for generating global cascades [212].

Different modifications of the original model [212] with different assumptions about the interpersonal influence and structure of the influence network yield different opinion dynamics but the general conclusions remain almost the same.

Networks of group structure. Real networks have a definite *local structure*. The authors [212] introduced a simple local structure in which acquaintances (friends) considerably influence each other and agents have multiple (often overlapping) groups of acquaintances. A population of n agents is partitioned into m groups of size g. In average each group is randomly connected with m_{avg} other groups. Each agent from group i is connected with each other agent from this group with some probability p and also with each other agent from m_i neighbor groups with some probability q. Two network structures are analyzed, *integrated* ($p = q$) and *concentrated* (in each group an agent has at least the same number of internal and external connections). As it turned out, such networks have a wider cascade window than the random networks considered earlier. However, group structure reduces the significance of opinion leaders, except for integrated networks corresponding to the left limit of the cascade window (the networks of low density). The same takes place for the role of early adopters: in sparse networks, of major importance are more influential adopters, and vice versa. Still these early adopters are not opinion leaders.

Change of influence rule. The assumption that many active neighbors are required for activating a leader seems rather reasonable. At the same time, it is interesting to study the setup in which leaders can be easily influenced like common agents. The paper [212] considered *the Susceptible–Infected–Removed* (*SIR*) *model* [103], a canonical model in which in each interaction an agent becomes active (in the terminology of epidemic models, *infected*) with some probability β and inactive (*recovers*) with some rate γ per unit time. In other words, most influential agents are easily influenced. In this case, the cascade window has no upper limit: the higher is network density (i.e., the susceptibility of all (!) agents also increases), the larger cascades occur. The general conclusions remain the same: as before, the cascades initiated by leaders are larger than the ones initiated by common agents, but without considerable difference. In average, the early adopters connected with initiators have higher influence than common agents; still, they are not opinion leaders.

1.1.3 Opinion Formation

In accordance with the definition of social influence, a given subject affects another subject in a social medium, changing his/her opinions (as well as emotions and behavior). This section describes in brief the existing publications on opinions dynamics in social networks using the mutual influence of agents.

In the first place, the fundamental research in this field is focused on the coordination models of agents' opinions (reaching consensus) in which the interaction of network members (social agents) gradually diminish the differences between their opinions. This phenomenon is explained by social psychologists using several factors such as conformity, acceptance of evidence (persuasion), incomplete awareness, self-distrust, etc.

The classical formal models of opinion dynamics (see [51, 66, 98] and also the surveys in [112, 180] as well as the opinion dynamics models in [33] and Sect. 2.1 of this book) consider sequential averaging for the continuous opinions of agents in discrete time. There exist different modifications of such models in which averaging runs in continuous time [1, 8], the opinions are measured in ordinal or even nominal scale, and so on. We will discuss a slightly varied example of the classical model (further referred to as the French–Harary–DeGroot model), which describes the dynamics of reaching consensus in a network structure. In this structure, at each step the nodes from the set $N = \{1, \ldots, n\}$ form their opinions as the weighted sum of the opinions of their neighbors and their own opinions at the preceding step. That is,

$$x_i^{(t+1)} = \sum_{j \in N} a_{ij} x_j^{(t)}, t \geq 0,$$

where $x_i^{(0)}$ denotes the opinion of agent i at some initial step; the parameter $a_{ij} \in [0, 1]$ reflects the influence level of agent j on agent i $\left(\sum_j a_{ij} = 1\right)$.

The opinion dynamics can be written in the matrix form

$$x^{(t+1)} = Ax^{(t)},$$

where A means the influence matrix, which is stochastic in rows.

These dynamics yield consensus in a strongly connected social network. The opinions of all agents are converging to the same value: each agent has direct or indirect influence on any other agent and the existing differences in their opinions are gradually vanishing. This model will be considered in detail in Chap. 2.

Note that the structure of influence networks imposes essential restrictions on the possibility of reaching consensus. Obviously, e.g., in an unconnected network consensus can be reached in special cases only. There may exist differing opinions in strongly connected networks, e.g., if the agents are less sensitive to influence [67]. In such models, at each step the agent's opinion is defined as the weighted sum of the opinions at the preceding step and the initial opinion:

$$x^{(t+1)} = \Lambda Ax^{(t)} + [I_n - \Lambda]x^0,$$

where $\Lambda = I_n - diag(A)$

The initial opinions of agents can be interpreted as their individual preferences or deep-rooted beliefs that have the same influence during opinions exchange.

Similar dynamics are observed in the opinion formation model [44] for the networks with compound nodes. In such networks, each node consists of two interacting agents, external and internal, and communicates with other nodes through his/her external agent only; the internal agent (treated as the confidant—friend or consultant—of the external agent) communicates with the latter only.

A multidimensional extension of the model with insensitive agents was described in [175]. Within this model, several interconnected issues (m different themes) are considered simultaneously, and each agent has some opinion on each of these issues. The opinion of agent i ($i \in N$) on m different issues is defined by the vector $x_i^{(t)} = (x_i^{(t)}(1), \ldots, x_i^{(t)}(m))$. At step t, the opinion vector of agent i has the form

$$x_i^{(t)} = \lambda_{ii} \sum_{j \in N} a_{ij} y_j^{(t-1)} + (1 - \lambda_{ii}) x_i^{(0)}$$
$$y_j^{(t-1)} = Cx_j^{(t-1)},$$

where C denotes the mutual influence matrix of the issues and $y_j^{(t-1)}$ are convex combinations of agent j in several issues. In matrix form the opinion dynamics can be written as

$$x^{(t)} = [(\Lambda A) \otimes C]x^{(t-1)} + [(I_n - \Lambda) \otimes I_m]x^{(0)}$$

where $\otimes$ means the Kronecker product, $\Lambda = I_n$ or $\Lambda = I_n - diagA$ depending on the model modification.

Generally speaking, within the above-mentioned models the mutual influence of agents gradually diminishes the difference in their opinions, even despite the consideration of additional factors (prejudice, common themes of interest) that preserve some divergence of opinions. In particular, by the hypothesis of averaging the opinions never leave a certain range of initial opinions.

So far, the dynamic models of opinion formation have been thoroughly studied on the theoretical issues of reaching network consensus. Such models are often analyzed using the apparatus of stochastic matrices, homogeneous and inhomogeneous Markov chains. As is well-known, opinion dynamics can be modeled by Markov chains. Consensus is reached in a homogeneous Markov network depending on the convergence of the power series of its stochastic matrix. Some sufficient conditions of such convergence were established in [18, 51]. For the stochastic matrices that do not guarantee consensus, the necessary conditions of consensus were suggested in [18] while the minimal variations of the initial opinions of agents (beliefs) leading to consensus were found in [2].

The dynamic model of agents' beliefs was further generalized in [41] so that the communication matrix varies at each step and the iterative process is defined by the product of matrices. The opinion coordination problem in this setup is reduced to the convergence analysis of inhomogeneous Markov chains. Moreover, this class of opinion formation models has close connection to the research of consensus in multiagent systems, another field of intensive investigations (e.g., see [38]). The theoretical results established there can be extended to social networks.

In the opinion dynamics models discussed above, consensus is reached through a gradual convergence of the initially different opinions of interacting agents to the same opinion. Besides consensus, real social networks also have other social and psychological phenomena such as group polarization (any initially dominating viewpoint becomes stronger in a group discussion) [149], opinions polarization (the disagreements between two opposing groups are deepening), non-conformism, etc. The classical models of opinion dynamics do not properly reflect the stable differences in opinions, clustering effects (the appearance of sets of agents with unique common opinions) or even the rise of radical opinions in highly connected network structures. Therefore, many researchers have been suggesting formal mathematical models that describe consensus together with other relevant factors (see [8, 52, 53, 101, 102, 114, 204] and Chap. 2 of this book). Particularly, the clustering of opinions as a macro-effect was considered in the bounded confidence models [52, 101, 102] under the assumption that sufficiently close agents (neighbors) may influence each other. This rule of interaction is often motivated by hemophilia and social identification.

1.1.4 Spread of Influence and Information

The influence of one subject on another in a socium affects the opinions and also behavior of the subject under influence. In this subsection, we will describe in brief some publications dedicated to the spread of activity (opinions, information, innovations) through social networks based on the influence of some agents on others. Linear threshold models and independent cascade models are among fundamental optimization and simulation models for information (activity) spreading in social networks.

Independent cascade models belong to the models of *interacting particle systems* and are closely related to the models of epidemics (also see the family of SI models).

In these models, the agents from a set $N = \{1, \ldots, n\}$ form a social network [75, 124]. Agents can be in two states, active or passive; denote by $S_t \subseteq N$ the set of active agents at a step $t \geq 0$. The weight of a graph edge (i, j) is the probability $p_{ij} \in [0, 1]$ with which agent i activates agent j.

The initial set $S_0 \subseteq N$ of all active agents at the step $t = 0$ is known. At each subsequent step $t \geq 1$, (a) all previously active agents at the step $t - 1$ preserve their state and (b) each agent $i \in S_{t-1} \backslash S_{t-2}$ activates one of his/her passive neighbors $j \in N \backslash S_{t-1}$ with the probability $p_{ij} \in [0, 1]$ independently of other agents.

The influence level of a set A of agents is defined as the expected number of active agents at the end of spreading given $S_0 = A$.

Models of threshold behavior. Whereas independent cascade models reflect the viral spread of some activity, the threshold models describe a complex form of agents' behavior in a social network [87] as follows. An agent becomes active only if the value of an aggregating function of all his positive signals (e.g., the sum of all signals or the number of such signals) exceeds a given threshold. Such threshold behavior is often called complex contagion [40]. As a matter of fact, there exist many mathematical models of threshold behavior [87, 124, 167].

A classical model is the *linear threshold model* [123, 124]. In this model, agents from a set $N = \{1, \ldots, n\}$ can be active or passive. Designate as $S_t \subseteq N$ the set of active agents at a step $t \geq 0$. The influence of agent j on agent i is defined by the weight $w_{ij} \in [0, 1]$ of the corresponding edge in the network graph. For an agent, the total influence of his/her neighbors satisfies the constraint $\sum_j w_{ij} \leq 1$.

The conservatism of agent i completely depends on his/her activation threshold $\phi_i \in [0, 1]$. In some models, all agents have the same fixed value ϕ_i (e.g., see [124]); other models proceed from the assumption that the threshold is random with some probability distribution [148]. In general, the individual differences of agents can be determined by their experience, convictions, personal traits, or the influence of mass media [205].

The linear threshold model has the following dynamics. There is a given set $S_0 \subseteq N$ of active agents at the initial step $t = 0$. At each subsequent step $t \geq 1$, (a) all previously active agents at the step $t - 1$ preserve their state and (b) each inactive agent $i \in N \backslash S_{t-1}$ becomes active if the influence of his/her active neighbors exceeds the activation threshold, i.e.,

$$\sum_{j \in S_{t-1}} w_{ij} \geq \phi_i.$$

The paper [124] suggested extensions for the linear threshold and independent cascade models as follows.

Generalized threshold model. Each agent v decides to be active in accordance with a monotonic activation function f_v: $S \subseteq N_v \to [0, 1], f_v(\varnothing) = 0$, where N_v is the set of his/her neighbors. So the activation function describes the local influence of neighbors. Each agent initially chooses a random threshold θ_v with the uniform distribution and then becomes active if $f_v(S) \geq 0$ (in addition, see [164]).

Generalized independent cascade model. The probability $p_v(u, S)$ that agent u activates agent v depends on the set S of agents who have already failed to do it. The model includes the following constraint: if neighbors $u_1, \ldots, u_l$ are trying to activate agent v, then the probability that he/she becomes active after l attempts is independent of the order of activation attempts.

As a matter of fact, the linear threshold model is not equivalent to the independent cascade model. However, the equivalence conditions of their generalized versions were established in [124]: for any generalized independent cascade model with given parameters, there exists an equivalent generalized threshold model, and vice versa.

Influence maximization in models of information spreading. As a rule, a subject applies informational influences to members of a social network for a maximal spread of necessary information (ideas, opinions, actions, innovations) through it. This goal can be achieved in different ways but often informational influences aim at a small number of key (influential) nodes.

The maximization problem for the spread of information (influence) has the following formal statement. Denote by S the initial set of active nodes—the initiators of information spreading in a network $G = (V, E)$. Let $\sigma(S)$: $2^V \to \mathbb{R}+$ be a known relationship between the expected number of active nodes in final analysis and the initial set of initiators (the function of global influence or resulting influence). By the budget constraint, it is possible to influence at most k nodes. The problem is to find a set S of k nodes that maximizes the function $\sigma(S)$. Also note other possible setups [42, 81, 141], with temporal spreading constraints: it is necessary to activate a given number of users in minimal time and/or using the minimal number of initiators. Alternatively, activation can be performed at different steps, not only at the initial step.

The paper [124] considered influence maximization for two basic models of spreading, namely, the linear threshold and independent cascade models. As demonstrated by the authors, this problem is *NP*-hard. The greedy heuristic $(1 - 1/e)$-optimal algorithm was also developed in [124] for choosing the initial set S. Influence maximization is similar to the maximization of submodular functions,[1] a well-known optimization problem that has been studied by several researchers (e.g., see [154] and modern surveys in the monographs [68, 201]). So the corresponding algorithms can be used after proving that $\sigma(M)$ is a submodular function. This was successfully done in [124].

However, submodularity is not guaranteed for global influence functions in some generalized threshold and independent cascade models; hence the greedy heuristic $(1 - 1/e)$-optimal algorithm becomes inapplicable for them. The following important result was established in [150]. Consider the generalized threshold models in which local influence functions f_v are monotonic and submodular; then the resulting global influence function $\sigma(\cdot)$ is also monotonic and submodular.

The greedy heuristic $(1 - 1/e)$-optimal algorithm suffers from poor scalability to large social networks, which forms its major drawback. A key component of this algorithm is calculation of influence level (global influence) for a given set of nodes, an $\#P$-hard problem [43, 208]. The influence level is estimated using the Monte Carlo simulations: activity spreading is executed very many times. Actually this greedy algorithm is $(1 - 1/e - \varepsilon)$-optimal, where ε depends on the number of executions (simulations) of activity spreading.

Many researchers suggested heuristics for a faster choice of the initial set S. Particularly, the heuristics introduced in [82] guarantee $(1 - 1/e)$-optimality with deferred calculations of the goal function (influence level) as follows. Owing to submodularity, at a current iteration of the greedy algorithm there is no need to calculate the goal function at a new node (candidate in the desired set) if the estimated increment of the goal function from adding this candidate at the previous iteration is smaller than its best increment from adding another node at the current iteration. Other heuristics are based on different approximations of the goal function and do not guarantee $(1 - 1/e)$-optimality [43, 83, 116, 117, 208].

The paper [124] also studied influence maximization in marketing. Consider m different marketing actions $M_1, \ldots, M_m$, each possibly affecting a certain subset of agents in a social network in order to increase the probability of activation. In other words, the initial set N_0 of active agents is not defined. An amount x_i is invested in each marketing action i. The total amount of investments satisfies the budget constraint. A marketing strategy forms a vector $\boldsymbol{x} = \{x_1, \ldots, x_m\}$. The probability $h_v(x)$ that agent v becomes active is determined by the strategy x. The function $h_v(\cdot)$ is nondecreasing and satisfies the diminishing returns property, i.e.,

$$\forall x \geq y, \forall a \geq 0 : h_v(x + a) - h_v(x) \leq h_v(y + a) - h_v(y).$$

As the result of direct marketing actions and further influence, the expected number of active agents makes up

[1] A submodular function f is a mapping of a finite set U into nonnegative real values that satisfies the diminishing returns property: the incremental output from adding an element to a set S is at least as high as the incremental output from adding this element to any set that contains S.

$$EL(\boldsymbol{x}) = \sum_{Z \subseteq V} \sigma(Z) \prod_{u \in Z} h_u(\boldsymbol{x}) \prod_{v \notin Z} [1 - h_v(\boldsymbol{x})].$$

Make the following assumptions for approximate maximization of this functional: (a) the value $EL(x)$ can be estimated at each point $\boldsymbol{x}$ and (b) it is possible to find direction i with almost maximal gradient. Denote by e_i the unit vector of axis i. For some value $\gamma \leq 1$, let there exist i such that $EL(\boldsymbol{x} + \delta e_i) - EL(\boldsymbol{x}) \geq \gamma\ (EL(\boldsymbol{x} + \delta e_j) - EL(\boldsymbol{x}))$ for any j, where δ is a constant. Then the approximate solution consists of two stages: (1) dividing the whole budget k into k/δ parts of amount δ and (2) at each step, investing δ in the marketing action M_i that maximizes the gradient of $EL(\cdot)$.

Influence minimization in models of information spreading. In many applications (particularly, informational safety), the spread of undesired information through a social network must be detected as soon as possible; in an alternative setup, the problem is even to minimize the spread of such information. To this effect, the states of a small group of social network nodes (*sensors*) are often monitored. In final analysis, the payoff of a control subject may depend on the spread detection time as well as on the number of spread cascades and the number of infected nodes that were successfully detected while his/her cost may depend on the properties of sensor nodes (see Sect. 3.2).

In [136], a social network was represented as a graph $G(N, E)$ in which the resources of a control subject are limited by a given value B. The following cascade spreading data were assumed to be available: each cascade initiated at node i reaches node u in a given time $T(i, u)$. The sensor set Z was found by maximizing the expected payoff, i.e.,

$$\max_{Z \subseteq V} R(Z) \equiv \sum_i P(i) R_i(T(i, Z)),$$

where $T(i, Z)$ denotes the minimal detection time for the cascade initiated at node i; P is the probability distribution of all cascades by their types (initiation nodes); $R_i(T(i, Z))$ specifies the payoff from detecting cascade i at the time $T(i, Z)$; finally, the cost function has the form $c(Z) = \sum_{a \in Z} c(a) \leq B$.

As shown in [136], the payoff functions are submodular in the sense that more sensors result in smaller marginal profit. Hence, the set Z can be calculated using the greedy algorithms [154].

Competing innovations. Up to this point, we have considered the models of single activity spreading in which an agent can be active or passive. In a more realistic scenario, several activities are spreading through a social network simultaneously (often similar competing innovations—ideas, opinions, products, etc.). The existing publications mostly deal with the spread of two opposite activities (innovations A and B, or positive and negative activities, see [95]), suggesting two types of models—the competitive independent cascade (CIC) and competitive linear threshold (CLT) models.

Consider the competitive independent cascade model in brief. In this model, each agent can be in one of three states (passive, positive active, or negative active). A passive agent may become active but the converse fails. Besides, an active agent cannot change his/her type of activity and also blocks the spread of the other type of activity (alien activity). At the initial step, all positively active agents form the set ${S_0}^+$ while all negatively active agents the set ${S_0}^-({S_0}^+ \cap {S_0}^- = \emptyset)$. Two probabilities are defined for each arc (u, v) of the graph, namely, $p^+(u, v)$ as the probability of positive activity transfer and $p^-(u, v)$ as the probability of negative activity transfer. At each step $t \geq 1$, all previously active agents are trying to activate their passive neighbors independently from each other with the probabilities defined on the graph arcs. In case of success, the activity type of an activated agent is determined by some rule (e.g., proportionally to the share of successful activation attempts of corresponding type). This process generates new sets of active agents, ${S_t}^+$ and ${S_t}^-$.

The resulting positive and negative influence functions, $\sigma^+({S_0}^+, {S_0}^-)$ and $\sigma^-({S_0}^+, {S_0}^-)$, are introduced by analogy with the corresponding classical models (the LTM and ICM, respectively).

The following setups of optimization problems arise accordingly.

In the first setup [32, 100], a control subject maximizes $(\sigma^-(\varnothing, {S_0}^-) - {}^-({S_0}^+, {S_0}^-))$ by choosing an initial set ${S_0}^+$ of cardinality k given the set ${S_0}^-$. In other words, he/she seeks to minimize the spread of alien activity (hazards, e.g., dangerous opinions or fake news) using the spread of his/her activity through the network (counterpropaganda).

In the second setup [22, 39, 142], a control subject maximizes the spread of his/her (positive) activity $\sigma^+({S_0}^+, {S_0}^-)$ by choosing an initial set S_0+ of cardinality k given the set S_0-. Here a possible interpretation is a known current spread of a competing product or news.

Consider this setup for the competitive independent cascade model in slightly modified notations. The paper [39] studied influence maximization for two competing innovations (players) A and B within the independent cascade model. Each agent in a network described by a graph $G(N, E)$ can be in one of three admissible states, A (adopted innovation A), B (adopted innovation B), and C (no decision yet). An agent may pass from state C to any other state; other state transitions are impossible. Denote by I_A and I_B ($I_A \cup I_B = I$) the initial nonintersecting sets of active nodes of corresponding types. For player A, the influence maximization problem is to maximize the expected number $f(I_A| I_B)$ of agents adopting innovation A given the set I_B with an appropriate choice of I_A.

The authors [39] suggested two extended versions of the independent cascade model, namely, *the distance-based model* and *the wave model*. In the former, each agent adopts the innovation from the nearest activated agent from I. The latter model describes step-by-step innovation spreading: a previously passive agent is activated at a current step by choosing uniformly a random neighbor at the distance proportional to step number.

As stated in [39], the functions are submodular, monotonic and nonnegative; moreover, approximation algorithms were developed for calculating the set I_A. The authors also noted that promising lines of further investigations are Nash equilibrium calculation and application of Stackelberg games.

These setups can be examined within the framework of hierarchical games in which one of the players chooses a best response to the action of the other. The main result here consists in influence maximization using greedy algorithms for best response calculation.

Other game-theoretic setups are also possible in which players choose their actions simultaneously and independently from each other. For example, the papers [7, 203] considered game-theoretic models of competing innovations (activities) with the following features. There exists a set of players and each member of this set is interested in a maximal spread of his/her activity through the network. Moreover, each player can influence the activity of network users at the initial step. This leads to a noncooperative normal form game, for which the authors established the existence of pure strategy Nash equilibria.

Models of competing innovations often neglect the fact that any online social network has an owner who controls it and may restrict any influences (e.g., some marketing campaigns). The paper [142] considered fair competitive viral marketing, i.e., a fair allocation of initial sets of users among the subjects seeking to preserve their activity in a social network.

Other Models of Activity Spreading

Note that most influence models discussed above involve simulations and their approaches are traditional for simulation modeling. They are close to collective behavior models [162], models of evolutionary games (e.g., see [206, 214]) and models of artificial societies, which are being intensively developed using agent-based simulation [132, 192].

Consider a series of influence models drawing some analogies with medicine, physics, and other sciences (also, see the survey [128]).

The models of percolation and contagion. These models have numerous applications—from epidemic simulation to oil field exploration—and represent a popular analysis method of information spreading. The classical *epidemic model* relies on the following cycle of contagion processes: initially, an individual is *susceptible* (S) to the disease; a contact with an infected individual makes him/her *infected & infectious* (I) with some probability β; after a certain period, he/she becomes *recovered/removed* (R) (i.e., restores the immune system or dies); with the course of time, immunity is reduced and he/she becomes *susceptible* again.

In *the SIR model* [10], a recovered individual is insusceptible to the disease, S $\rightarrow$ I $\rightarrow$ R. The whole society consists of three groups, $S(t)$ (the individuals who are not infected yet or are susceptible to the disease at a time t), $I(t)$ (the infected individuals), and $R(t)$ (the recovered individuals). Let $N = Const = S(t) + I(t) + R(t)$. The epidemic dynamics are described by the following system of three differential equations: $\frac{dS(t)}{dt} = -\beta\, N \frac{S(t)}{N} I(t) = -\beta\, S(t)\, I(t)$ (i.e., each individual infected per unit time is contacting susceptible individuals and makes them infected with the probability β); $\frac{dR(t)}{dt} = \gamma I(t)$ (i.e., infected individuals are recovering from the disease after the average period 1/γ); finally, $\frac{dI(t)}{dt} = \beta S(t)I(t) - \gamma I(t)$.

There exist other, more complex models of epidemics (see the surveys in [112, 155]), particularly, *the SIRS model* in which a recovered individual becomes susceptible with the course of time. The flu is a natural example of disease well described by this model. Another phenomenon fitting this model is information spreading in a social network. A blogger reads a blog of his friend (susceptible) dedicated to some theme, then writes his/her own blog on it (infected), getting back to this theme later (susceptible).

For social networks, a crucial role is played by *the epidemic threshold* λ_c, an index that defines the critical probability of infection for neighbors. Once the epidemic threshold is exceeded, infection spreads through the whole network. This index depends on the properties of social network graph, e.g., on the number of nodes, the distribution of connections, the coefficient of clustering, and others. So infection spreading strongly depends on the representation model of network graph.

If a social network is described by a random graph, then infection with a probability exceeding the epidemic threshold has exponential growth: $\lambda = \beta/\gamma > \lambda_c$. On the other hand, infection with smaller probability is vanishing with exponential rate.

A more realistic model of a social network is a *scale-free graph* in which some nodes have connections to thousands or even millions of other nodes that are *par excellence* connected to a few nodes (no characteristic scale). In such graphs, the number of node connections satisfies the power distribution [14]. As indicated by the spread analysis of computer viruses [183], scale-free networks have no epidemic threshold: in case of infection, the epidemic surely spreads through the whole network. However, in social networks the themes of discussion may spread without epidemics and hence the threshold is actually nonzero. So a more adequate model is required for the networks with the power distribution that would reflect the delicate properties of such networks (e.g., the coefficient of clustering [58]). An alternative approach consists in an appropriate modification of the infection transfer model for reducing the probability of infection for larger "distances to an initiator" [215].

Avalanche-like processes. As noted in [50], a wide range of natural and social phenomena (combustion and explosion, the replication of viruses or the accumulation of decay products in a living organism, social conflicts [221] induced by meeting processes, stockjobbing and agiotage for different products, the diffusion of technological and managerial innovations including information systems and technologies, informational influences on individual and collective subjects) has a common distinctive feature that unites all these processes and systems into the same class. The matter concerns their avalanche-like spread and evolution (similarly to chain reactions) and consequently the presence of internal or external connections with a large (often exponential) variation of one parameter under small variations of the other. In [50] such processes were called *fast social and economic processes*. The cited monograph contains a rich spectrum of models for these processes (as well as the models of percolation and contagion and cellular automata models) and also results on their simulation and identification using real data.

Cellular automata models. The processes of information spreading in social networks can be described within the following approach. Consider a social network as a complex adaptive system that consists of very many agents interacting with each other. Such interactions form some collective behavior, which is difficult to predict and analyze. This class of complex systems can be studied and modeled using cellular automata. A cellular automaton (e.g., see [188]) is a set of objects (here, agents) often representing a regular lattice. At each discrete time, the state of a separate agent is characterized by some variable. All states are synchronously evolving in discrete intervals in accordance with fixed local probabilistic rules, which may depend on the state of a given agent and also on the states of his/her *nearest* neighbors.

The paper [75] modeled *the word-of-mouth effect* in information spreading through social networks. Each agent in a large network belongs to a single personal network in which agents have *strong* (fixed and stable) *connections*. Each agent also has *weak connections* to the agents from other personal networks (weak and strong connections were discussed in [86]). The probability that at a given time an informed agent influences an uninformed one via a strong connection (making him/her informed) is β_s; via a weak connection, $\beta_w < \beta_s$. Besides, at a given time uninformed agents become informed with a probability α as the result of advertising and other marketing tricks. (In accordance with empirical evidence [37], this probability is smaller in comparison with the word-of-mouth effect.).

Thus, at each time t, an uninformed agent having m strong connections to informed agents from his/her personal network and j weak connections to informed agents from other personal networks becomes informed with the probability

$$p(t) = (1 - (1 - \alpha)(1 - \beta_w)^j(1 - \beta_s)^m).$$

The authors [75] suggested the following probabilistic cellular automaton.

1. All agents are originally uninformed (state 0).
2. At the initial time, the agents become informed by advertising because spreading based on the word-of-mouth effect needs informed agents. For each agent, a random number U ($0 < U < 1$) is generated and compared with the probability $p(t)$ of informing. If $U < p(t)$, then this agent becomes informed (state 1).
3. At the subsequent times, the word-of-mouth effect with strong and weak connections is activated. Again, if $U < p(t)$, then an agent becomes informed (state 1).
4. The process repeats until 95% of agents become informed.

The following parameters were specified in [75] for simulations: the size of each personal network; the number of weak connections for each agent; the probabilities β_s, β_w, and α. As it turned out, despite smaller probability the weak connections have at least the same impact on the rate of information spreading as their strong counterparts. For the initial phase (*early informed*), the awareness of agents considerably depends on advertising but its role gradually diminishes. For the next phase (*middle informed*), information spreads through personal networks via strong connections; as the number of informed agents in such networks is increasing, the impact of strong connections is vanishing while the role of weak connections in the activation of new networks is rising. For larger personal networks, the role of strong connections is increasing while that of weak connections decreasing. In the networks with many weak connections, the effect from strong connections becomes lower (from weak connections, higher). Under intensive advertising campaigns, the effect from strong connections is slightly increasing while that of weak connections decreasing.

Model of marketing actions based on Markov network. Consider a social network as a set of potential customers of some product or service or potential adopters of a new technology (innovation). From the supplier's viewpoint, *the agent's value* (utility) in a social network depends on this agent (e.g., the expected profit from selling the product or technology to him/her) and also on his/her influence on other agents. In other words, of crucial importance are the configuration and state of a social network—the opinions of potential customers about the product (see examples in [108]). So it is necessary to identify a small number of agents (e.g., for suggesting preferential terms of sales), who would facilitate the spread of innovation through the whole network. Actually, this is an *influence maximization* problem.

The authors [57] also addressed the problem of k most influential agents in a social network, with application to *viral marketing*. A market was modeled by a social network of agents (Markov network) in which the value of each agent depends on the expected profit from selling a product to this agent (*the intrinsic value of customer*) and also to other agents under his/her influence, and so on (*the network value of customer*).

In [57], the problem of optimal marketing actions $MA = \{MA_1, \ldots, MA_n\}$ was stated as follows. Denote by MA_i a marketing action associated with agent i (Boolean variable, 1—discount and 0—no discount, or continuous variable that defines the amount of discount). For a set of n agents, let the predicate $X_i = 1$ if agent i buys the product and $X_i = 0$ otherwise. Assume the product possesses several attributes $Y = \{Y_1, \ldots, Y_m\}$. For each agent i, there is a set N_i of his/her neighbors affecting X_i (the network of agents). In turn, agent i has influence on his/her neighbors.

Introduce the following notations: c as the marketing cost per one agent; rv_1 as the sale proceeds from an agent with a marketing action; rv_0 as the sale proceeds from an agent without a marketing action. If a marketing action includes a discount, then $rv_1 < rv_0$; otherwise, $rv_1 = rv_0$. For the sake of simplicity, consider the Boolean vectors MA only.

Denote by $f_i^1(MA)$ the result of making $MA_i = 1$ provided that all other elements of this vector are invariable and define $f_i^0(MA)$ by analogy. Then the expected profit increase from a marketing action (for a single agent without the influence on other agents), i.e., *the intrinsic value of agent i* is calculated by

$$ELP_i(X^k, Y, MA) = rv_1 P(X_i = 1|X^k, Y, f_i^1(MA)) - rv_0 P(X_i = 1|X^k, Y, f_i^0(MA)) - c$$

where X^k gives the set of agents with known states (who surely purchased the product or not) and $P(X_i \mid X^k, Y, MA)$ is the conditional probability of product purchase for agent i.

Consequently, for selected agents the total expected profit increase from all marketing actions makes up

$$ELP(X^k, Y, MA) = \sum_{i=1}^{n} rv_i P(X_i = 1|X^k, Y, MA) - \sum_{i=1}^{n} rv_0 P(X_i = 1|X^k, Y, MA_0) - |MA|c$$

where MA_0 denotes a zero vector; $rv_i = rv_1$ if $MA_i = 1$ ($rv_i = rv_0$ otherwise); finally, $|MA|$ is the number of selected agents.

The total value of agent i is defined by $ELP(X^k, Y, f_i^1(MA)) - ELP(X^k, Y, f_i^0(MA))$. (That is, the value MA changes for other agents and may affect their probabilities of purchase). Then *the network value of each agent* is the difference between his/her total and intrinsic values. So the value of each agent depends on the marketing actions carried out for other agents and also on product purchases by them.

Again consider the problem of k most influential agents in a social network. Obviously, such agents can be identified by calculating an action MA that maximizes ELP. In the general case, the exhaustive search of all possible combinations is required to find the optimal action MA. The following approximation procedures yield almost optimal solutions.

(1) Single pass. For each agent i, set $MA_i = 1$ if $ELP(X^k, Y, f_i^1(MA_0)) > 0$, and set $MA_i = 0$ otherwise.
(2) Greedy search. Let $MA = MA_0$. Loop through all MA_i and set $MA_i = 1$ if $ELP(X^k, Y, f_i^1(MA)) > ELP(X^k, Y, MA)$. Continue looping until there are no changes in a complete scan of the MA_i.
(3) *Hill-climbing search*. Let $MA = MA_0$. Set $MA_{i_1} = 1$, where $i_1 = \arg\max_i (ELP(X^k, Y, f_i^1(MA)))$. Now set $MA_{i_2} = 1$, where $i_2 = argmax_i(ELP(X^k, Y, f_i^1(f_{i_1}^1(MA))))$. Repeat this until there is no agent i such that $MA_i = 1$ increases ELP.

Influence models based on Bayesian networks. The paper [219] suggested an influence model for a group of agents (team) that represents a *dynamic Bayesian network* (*DBN*) with the following two-level structure. The first level (individuals) is used to model the actions of each agent while the second level (group) the actions of the whole group. There are N agents totally. At each time t, (a) agent i is in a state S_t^i whose probability $P(S_t^i|S_{t-1}^i, S_{t-1}^G)$ depends on the agent's previous state and the state of the whole group; (b) agent i performs an action O_t^i with the conditional probability $P(O_t^i|S_t^i)$; (c) the group is in a state S_t^G whose probability $P(S_t^G|S_t^1,\ldots,S_t^N)$ depends on the states of all agents. Therefore, the probability that at some time T the group of N agents performs the aggregate action O in the aggregate state S makes up

$$P(S,O) = \prod_{i=1}^{N} P(S_1^i) \prod_{i=1}^{N}\prod_{t=1}^{T} P(O_t^i|S_t^i) \prod_{t=1}^{T} P(S_t^G|S_t^1,\ldots,S_t^N) \prod_{t=2}^{T}\prod_{i=1}^{N} P(S_t^i|S_{t-1}^i S_{t-1}^G).$$

Introduce a new variable Q that defines group state, and make the following assumptions.

(a) This variable is independent of the states of other agents.
(b) When $Q = i$, the group state S_t^G depends on the state S_t^i of agent i only.

Then $P(S_t^G|S_t^1,\ldots,S_t^N)$ can be written as

$$\sum_{i=1}^{N} P(Q=i)P(S_t^G|S_t^i) = \sum_{i=1}^{N} \alpha_i P(S_t^G|S_t^i),$$

where α_i denotes the influence of agent i on the group state.

The described two-level influence model has close connection to a series of other models, namely, *the mixed-memory Markovian model* (*MMM*) [187], *the coupled hidden Markovian model* (*CHMM*) [171], and *the dynamical systems trees* (*DST*) [106]. *The MMM* decomposes a complex model (e.g., a Markovian model of order K) in the following way:

$$P(S_t|S_{t-1}, S_{t-2},\ldots,S_{t-K}) = \sum_{i=1}^{K} \alpha_i P(S_t|S_{t-i}).$$

The *CHMM* describes the interaction of several Markov chains through a direct connection between the current state of one stream and the previous states of all other streams: $P(S_t^i|S_{t-1}^1, S_{t-1}^2,\ldots,S_{t-1}^N)$. However, this model is computationally intensive and hence often simplified as follows:

$$P(S_t^i|S_{t-1}^1 S_{t-1}^2,\ldots,S_{t-1}^N) = \sum_{j=1}^{N} \alpha_{ji} P(S_t^i|S_{t-1}^j),$$

where α_{ji} denotes the influence of agent j on agent i. The suggested model extends these models using the variable S^G of group level, which reflects the influence between all agents and the group $P(S_t^G|S_t^1,\ldots,S_t^N) = \sum_{i=1}^N \alpha_i P(S_t^G|S_t^i)$ as well as defines the dynamics of each agent depending on the group state $P(S_t^i|S_{t-1}^i, S_{t-1}^G)$. DST have a tree structure that models the interactive processes through hidden Markov chains. There exist two distinctions between the *DST* and the model discussed above [219]. First, in *the DST* the predecessor has its own Markov chain; in the other model, the current state of the group does not directly depend on its previous state (i.e., an action of the group is the aggregated action of agents). Second, in the model [219] the group and agents affect each other, in contrast to *the DST*.

The authors [219] expected that their multilevel influence approach would be a good tool for social dynamics analysis and revelation of group behavior templates.

Voter models. The stochastic voter model that belongs to the class of interacting particle systems was suggested by Clifford and Sudbury [49, 139]. Its original interpretation was associated with voting on two political options. Nevertheless, the voter model can be used to describe the spread of opposite opinions in social networks [61, 138, 176].

Influence maximization based on the voter model was considered in [61]. In this model, a social network is represented by an undirected graph with loops $G(N, E)$. Each node $v \in N$ has a set of neighbors $N(v)$ and arbitrary initialization with value 1 or 0. At a time $t + 1$, node v chooses one of the neighbors (with the same probability) and adopts his/her opinion:

$$f_{t+1}(v) = \begin{cases} 1, & \text{with the probability } \frac{|\{u \in N(v): f_t(u)=1\}|}{|N(v)|}, \\ 0, & \text{with the probability } \frac{|\{u \in N(v): f_t(u)=0\}|}{|N(v)|}. \end{cases}$$

This model is similar to the classical linear threshold model in the sense that with higher probability each agent changes his/her opinion to the opinion adopted by most of his/her neighbors. Unlike the threshold model, in the voter model each agent may pass to the passive state.

Let c_v be the initial persuasion cost for agent v ($f_0(v) = 1$). The influence maximization problem has the following statement: find f_0: $N \to \{0, 1\}$ that maximizes the expected value $E[\sum_{v \in N} f_t(v)]$ under a given budget constraint $\sum_{\{v \in N | f_0(v)=1\}} c_v \leq R$. As established in [61], in the case of homogeneous agents with the same initial persuasion cost, the optimal solution is to select k network nodes with highest degree. This result well matches the heuristic often used in practice.

Influence models based on Ising model. The Ising model is a mathematical model that describes ferromagnetism [129]. (Note that this model is treated as an example of Markov networks.) It considers the interaction of nearest atoms-neighbors in a crystal lattice; so there exists an obvious analogy with the relations of neighbor agents in a social network. The interaction energy is given by $E_{ij} = -J\, S_i\, S_j$, where s denotes atomic spin (± 1) and J means the interaction constant. The total energy $E(S)$ can be calculated by summing over all atoms of the lattice: $E(S) = -J\sum_{i\sim j} S_i\, S_j$. The total energy in an external field h has the form $E(S) = -J\ \sum_{i\sim j} S_i\, S_j + h\ \sum_i S_i$.

For a ferromagnetic, the interaction constant is $J > 0$, and the energy takes minimal value for the codirectional spins. The system entropy is minimal in the ordered state (under minimal energy) and demonstrates fast growth with the system energy. For temperatures below the critical threshold, almost surely most of the atomic spins have the same orientation; for higher temperatures, their orientation is random.

As hypothesized in [199], conformity or independence in a large social group can be well described by the Ising model. Nearest neighbors have determinative influence while the group's capability to think creatively and adopt new ideas is an analog of temperature. Finally, the role of an external field in a social group is played by authority or control.

1.2 Common Knowledge. Collective Actions

1.2.1 Role of Awareness

Consider an agent that enters some social network. An agent is informed about current *situation* (the actions and beliefs of other agents, the parameters of an external environment—the so-called *state of nature*, etc.). Situation determines the agent's set of values and beliefs and his/her attitude that are interconnected in the following way: values affect attitude; in turn, the latter produces inclinations toward certain beliefs; these inclinations are associated with a hierarchical system of beliefs about the environment that exists in the agent's memory.[2] Inclinations toward certain beliefs and current situation (e.g., the actions of other agents) form new or change old beliefs. Being guided by these beliefs and a preset goal, an agent makes a decision and acts accordingly. The results of actions change current situation as well as the internal values, beliefs and attitude.

Beliefs of order *n*. Mutual beliefs. For agent's decision-making, of crucial importance are his/her beliefs about the beliefs of other agents and so on. "Agent A knows that agent B knows that agent C knows the value p" is an example of beliefs of the second order (the second *reflexion rank*). Really, before acting each agent tries to predict the behavior of other agents. At the same time, other agents may have their own beliefs of different orders (see the survey and models of mutual beliefs in [168]).

For many social relations, events and actions in which participants have no explicit agreements or contracts, a major role is played by the mutual beliefs based

[2]Note that the terms "value," "belief," "attitude," etc. have different interpretations in modern psychology and theory of multiagent systems that do not match each other.

on the identity of their beliefs (see *conventions* in [137, 168]). Two approaches to introduce the concept of mutual beliefs were separated out in [202] as follows.

(I) *Iterative approach*. In accordance with this approach, there exists a mutual belief about p in a group M if and only if (a) all agents from M know p; (b) each agent knows that all agents know p; (c) and so on, ad infinitum. (In other words, the fact p is *common knowledge* for all agents from M [168].) This approach relies on the hypothesis that the agents have beliefs of very high reflexion ranks. In reality, this is impossible or the agents simply cannot handle such beliefs due to limited cognitive capabilities, lack of information or insufficient rationality. (The problem of *maximal reasonable reflexion ranks* is called *the level problem* in the Western literature). There exist several ways to solve this problem (also see [168]).

(a) An agent in a group may act without distrusting in the judgement p (*the lack of disbelief*, which is defined as no beliefs about the negation of p);
(b) All agents in a group merely have inclinations towards the beliefs of higher order; in this case, the agents must be properly informed and possess common reasoning rules for making the same conclusions.

In both cases, the agents must be rather rational and intelligent for obtaining the beliefs of higher order without difficulty (if necessary). A possible prerequisite for this process is what the beliefs of higher order actually suggest. Successful actions often rely on the beliefs of the second order (generally speaking, this is not the case —see the sufficient conditions in [168]).

Example 1.1 (an action performed jointly by two agents). The judgements of the basic order are as follows.

(1) Agent A will perform his/her part of the action X, $p(A)$.
(2) Agent B will perform his/her part of the action X, $p(B)$.

Make the following assumptions.

(i) A expects (1) and (2).
(ii) B expects (1) and (2).

As established in [202], the level $n = 2$ is necessary and sufficient for performing the joint action in this example. Obviously, agent A must be expecting that agent B will perform the part $p(B)$ of the action: in this case, the former is willing to perform his/her own part. Agent A must be also expecting that agent B believes that agent A will perform the part $p(A)$. Otherwise, there are no sufficient grounds for agent A to expect that agent B will perform the part $p(B)$. (If, by the assumption of agent A, agent B is not expecting that the former will perform $p(A)$, then he/she will not perform $p(B)$ too.) So, in this case agent A will not perform his/her part. The same reasoning applies to agent B. Consequently, it is necessary that the agents have beliefs of the following order.

(iii) A expects (i) and (ii).
(iv) B expects (i) and (ii). •[3]

Example 1.2 Assume that each member of the Society of Flat Earth knows that the Earth is flat and, in addition, that all other members know it (because they belong to this society). Each agent recognizes that the other agents in the group believe in the same thing, and this establishes a social relation among them based on belief. But this recognition disappears at the first level. Again, the second level is necessary (and often sufficient) for the coordinated decisions of the members on the Earth shape. Let somebody find out that they all have the same belief of the second order. Then this person may study the possible beliefs of the third order, etc. •

Note that some social concepts are independent of mutual belief, e.g., hidden social influence and power.

(II) *Reflexive approach*. There exists a mutual belief about p in a group G if and only if (a) all agents from G assume p; (b) all agents assume there is a mutual belief about p in G. This corresponds to the second reflexion rank [168].

Mutual belief as shared "we-belief." "We-belief" is an agent's belief about p that satisfies the following properties.

(a) Each agent has this belief and assumes that
(b) all agents in a group have the same belief and also each agent assumes that
(c) all agents assume there is a mutual belief in the sense of (b).

Property (a) is required since there may exist no shared position without all agents. Property (b) provides social cause for adopting this position. Property (c) strengthens the cause, making it intersubject.

A shared "we-belief" implies that each agent has a group of "we-beliefs," i.e., matches the third level of reflexion [168].

Mutual beliefs can be used to explain collective thinking and collective actions [162, 168]. Some models with strategic and informational reflexion of social network members will be described in Sect. 2.4 of this book.

1.2.2 Public Goods and Specialization

Collective actions, public goods, and coordination games. The following factors are important for *collective actions*: awareness, communication, and coordination. *Collective action theory* well describes a wide range of phenomena (public movements, electorate behavior, membership in interest groups) connected with the achievement of public goods through coordinated joint actions of two or more

[3]Throughout the book, • denotes the end of an example.

individuals. This theory also studies the influence of external factors on group behavior.

As is well-known, *public goods* are characterized by

(1) guaranteed consumption, i.e., it is impossible to eliminate those consumers of public goods who have not paid for them;
(2) no competition during consumption: the fact of public goods consumption by one subject does not reduce the consumption for other subjects.

Pure public goods are national defense, bridges, public opinion, open-access databases, communication systems, etc.

For achieving the same public good (goal), two or more individuals make a *collective action*. Each individual chooses between *participation* and non-participation (*free riding*) in a collective action. Participation in a collective action incurs some individual cost for achieving public goods, and each individual can potentially gain without any cost (for a large number of individuals, the public cost to detect free riders and impose sanctions increases). This leads to obvious difficulty in the implementation of mutually beneficial collective actions, *the free rider problem* well-known in modern microeconomic theory (e.g., see [144, 151]).

Agents can be stimulated to participate in public goods production through incentives or certain social pressure schemes. As emphasized in [145, 172], only *organizations* may bear cost for solving these tasks (organizations play key role in interaction, motivation, communication and coordination of collective activity [162, 165]). At least latent groups are needed, i.e., communities of common group interest in public goods that still exist without organizational structure for communication and organization but with structural leadership (Principals) for resource accumulation and decision-making [145].

However, recent advances in information and telecommunication technologies (personal computers, mobile phones, e-mail, chatting, Internet) for collective actions have dramatically reduced the cost of communication and coordination [20], in some cases eliminating the need for formal structure design. For example, consider information of public utility as a public good. For the purposeful creation of such a database and corresponding community, the coordination of participants is required at the initial stage, which leads to the free rider problem. If such a database is created in a purposeless way (i.e., participants do not know other participants and publish their information independently at different public sources—forums, web pages), *trust* rather than *participation* is the major problem.

Collective action situations in which all parties may gain from mutually coordinated decisions (the coordination problem) are often modeled by coordination games. *Coordination games* form a class of games with multiple pure Nash equilibria in which players choose the same or coordinated strategies [80, 99, 107, 153, 168]. In this book, we will consider collective actions of different agents within social networks.

Collective actions in social networks. Here a key role is played by social relations. On the one hand, social bonds can guarantee an efficient local social control for stimulating a wide participation in collective actions (as the result of social pressure from neighbors, trust, social approval, the need for keeping positive relations and meeting expectations, anaclisis, reputation maintenance, identification with neighbors, etc.). For instance, the behavior of a given agent depends on the actions of his/her neighbors. On the other hand, social bonds provide a given agent with information about the intentions and actions of other agents in a network and also form his/her (incomplete) beliefs for further decision-making. Finally, within the limits of social bonds agents can put joint efforts to produce local public goods and utilize them together. So the structure of a social network strongly affects the agents' decision-making and participation in collective actions.

The paper [46] considered the following game-theoretic model of collective actions in a network. Agents are putting joint efforts to some collective action, which is successful (yields a positive contribution to the goal functions of agents) if the total effort exceeds a given threshold. The higher is the agent's effort in the successful action, the greater is his/her payoff. In addition, the agent's effort itself makes some negative contribution to his/her goal function, which depends on the agent's type (also called efficiency): the larger is the type, the less effort the agent needs to put (particularly, in psychological terms this can be explained by higher loyalty and sympathy to the collective action). The authors [46] established a condition for the number of agents and their efficiencies under which nonzero actions are an equilibrium; moreover, they described possible influence on the agents' types that implements the zero action (no actions for all agents) as a unique equilibrium.

Public goods in social networks. The paper [23] studied *public goods* in a social network as follows. Agents with social bonds are putting joint efforts to produce public goods and utilize them. As noted in [23], this may cause network *specialization* subject to public goods (also see the surveys in the theory of *network games* [77, 113]). It was proved in [23] that there exists an equilibrium in which some agents are making contribution (efforts) while the other take advantage of it. Such specialization can be fruitful for the whole society if the contributors (specialists) have many connections in the network. New connections in the network improve the availability of public goods yet reduce individual motivations for additional efforts (higher contribution). Consequently, the total welfare is greater in incomplete networks. In this sense, a promising line of further research can be the dynamics and formation processes of social networks.

The authors [23] introduced a network model of public goods with a fixed network structure. (A connection between agents i and j is defined by a binary value $r_{ij} = r_{ji}$). Agent i chooses the amount e_i of his/her efforts for putting in a public good, which will be utilized by all his/her neighbors from a set N_i. The agent's payoff is described by a twice differentiable strictly concave function $f(\cdot)$ that depends on the *efforts* (actions) e_i of the agent and also on the efforts $\overline{e}_i$ of his/her neighbors. So the payoff function is defined by the effort vector $\boldsymbol{e}$ of agents and by the graph G: $U_i(\boldsymbol{e}; G) = f(e_i + \sum_{j \in N_i} e_j) - c\, e_i$, where c gives the cost of unit effort.

The game considered in [23] has a given network structure G and the agents are simultaneously choosing their efforts to maximize their payoff functions. Due to obvious symmetry of this game, for all agents the payoff functions are maximized by the same effort e^* that satisfies $f'(e^*) = c$. A vector $\boldsymbol{e}$ is *a Nash equilibrium* if and only if, for any agent i, either $\overline{e}_i \geq e_i^*$ and $e_i = 0$ (agent i puts no effort at all), or $\overline{e}_i < e_i^*$ and $e_i = e_i^* - \overline{e}_i$ (some effort).

Positive and negative effects from new connections. On the one hand, a new connection improves access to public goods; on the other, destimulates an agent to put his/her efforts. Denote by $W(\boldsymbol{e}, G)$ the total payoff of all members of a social network G in an action vector $\boldsymbol{e}$. An action vector $\boldsymbol{e}$ is called *second-best* (or *utilitarian* in the terminology of [151]) if there does not exist another vector $\boldsymbol{e}'$ such that $W(\boldsymbol{e}', G) > W(\boldsymbol{e}, G)$. In the general case, second-best vectors are defined subject to some constraints. Consider a second-best vector $\boldsymbol{e}$ in a graph G in which agents i and j are not connected. Add the new connection (i, j). As a result, if agent i put no effort before, the equilibrium remains the same and hence $W(\boldsymbol{e}, G + ij) > W(\boldsymbol{e}, G)$; if both agents put some efforts, then the equilibrium $\boldsymbol{e}$ is replaced by a new equilibrium, possibly with smaller public welfare.

The authors [23] introduced the following modifications in the model.

(1) *Imperfect substitutability of efforts*: $e_i + \delta \sum_{j \in N_i} e_j$, where $0 < \delta \leq 1$ (the agent's effort yields higher individual payoff than the efforts of other agents). If the value δ is sufficiently small, then the agents are putting strictly positive efforts and there exists a unique *distributed* Nash equilibrium (no specialization). As proved in the paper, a *specialized vector* is an equilibrium if and only if non-specialists are connected at least with $s = 1/\delta$ specialists from a maximal independent set.

(2) *Convex costs*. Let the effort cost $c(e_i)$ be an increasing convex function that satisfies the condition $c'(0) > f'(+\infty)$. In this case, the agent benefit from joint efforts. In a complete graph, there exists a unique equilibrium with a distributed action vector (i.e., all agents are putting the same efforts). Specialization is possible in incomplete graphs. An effort $\boldsymbol{e}^*$ that maximizes the payoff function is achieved under the condition $f'(\boldsymbol{e}^*) = c'(\boldsymbol{e}^*)$. Assume an integer s is such that $f'(s\,\boldsymbol{e}^*) \leq c'(0)$. It was established that a specialized vector forms an equilibrium if and only if non-specialists are connected with at least s specialists from a maximal independent set.

(3) *Heterogeneous agents*. Let each agent i have individual cost c_i and payoff f_i functions as well as a specific effort e_i^* that maximizes his/her payoff. A vector $\boldsymbol{e}$ is a Nash equilibrium if and only if, for any agent i, either $\overline{e}_i \geq e_i^*$ and $e_i = 0$ (agent i puts no effort), or $\overline{e}_i < e_i^*$ and $e_i = e_i^* - \overline{e}_i$ (some effort). In a complete graph (to say nothing of incomplete ones), there exists a unique Nash equilibrium in which agents with high threshold are putting nonzero effort (in other words, heterogeneity leads to specialization).

1.2.3 Communication and Coordination

The paper [47] considered a social network as a communication network through which agents are informing each other about their willingness (decision) to participate or not in a collective action. Each agent knows the decision of his/her nearest neighbors; based on this local knowledge, each agent makes his/her own decision in accordance with the following rule: "I will participate if you do" (*the coordination mechanism*). This setup leads to a coordination game with incomplete (imperfect) awareness. *Communication networks* facilitate coordination, and of major interest are their properties that allow for collective actions. The authors [47] studied minimal sufficient networks in which all agents are arranged in *a hierarchy of social roles*/stages—*initial adopters*, *followers*,…, *late adopters*. Such networks facilitate coordination in the following ways:

(1) each level is informed about the earlier levels;
(2) common knowledge is generated at each level.

So the role of (locally) common knowledge in a collective action is identified and a correlation between the structure of a given social network and common knowledge is established.

In the model [47], a group of agents is described by a finite set $N = \{1, 2, \ldots, n\}$. Each agent $i \in N$ is willing (w) to participate in a collective action or not (x). The admissible states of nature form the set $\Theta = \{w, x\}^n$. Each agent i makes a decision $a_i \in \{r, s\}$, where r means participation and s non-participation. The agent's payoff depends on his/her willingness to participate and also on the decisions of all other agents in the group. By assumption, his/her utility function u_i: $\{w, x\} \times \{r, s\}^n \to \mathfrak{R}$ possesses the following properties:

$$u_i(x, a) = \begin{cases} 0 \text{ if } a_i = r, \\ 1 \text{ if } a_i = s \end{cases} \tag{1.1}$$

(an agent always stands aside if he/she is not willing to participate);

$$\begin{aligned} \forall a, a' \in \{r, s\}^n \text{ such that } a'_j = r \Rightarrow a_j = r : u_i(w, r, a_{N\backslash\{i\}}) \\ -u_i(w, s, a_{N\backslash\{i\}}) \geq u_i(w, r, a'_{N\backslash\{i\}}) - u_i(w, s, a'_{N\backslash\{i\}}) \end{aligned} \tag{1.2}$$

(the agent's willingness to participate increases with the number of other agents participating in the collective action). In other words, the utility function is supermodular, see the definition above.

However, also it is necessary to consider the social network itself (the communication network in which social bonds determine the directions of information transfer). Each agent knows about the existence of all other agents in the network (this fact is common knowledge). The network describes binary relations $\to$ over the set N: the relation $j \to i$ means that agent i is informed about the intentions of agent j. Agent i knows only his/her own intentions and also the intensions of his/her neighbors $B(i) = \{j \in N \mid j \to i\}$. So agent i merely knows that the real state of nature $\theta \in \Theta$ belongs to the set of indistinguishable states of nature $P_i(\theta) = \{(\theta_{B(i)}, \varphi_{N \backslash B(i)}) | \varphi_{N \backslash B(i)} \in \{w, x\}^{n-\#B(i)}\}$. Such states form $\Im_i = \{P_i(q)\}_{\theta \in \Theta}$, i.e., *an informational partition* of all admissible states Θ for agent i.

The strategy of agent i is a function $\boldsymbol{f}_i$: $\Theta \to \{r, s\}$ such that, for any $\theta, \theta' \in \Theta$, the equality $f_i(\theta) = f_i(\theta')$ holds if $\theta, \theta' \in P \in \Im_i$. In other words, agent i cannot discriminate between two states belonging to the same element of the partition $\Im_i$, and he/she makes the same decision accordingly. Denote by F_i the set of all strategies of agent i, $F = \prod_{i \in N} F_i$.

The expected utility of agent i for $f \in F$ is defined by $EU_i(\boldsymbol{f}) = \sum_{\theta \in \Theta} \pi(\theta) u_i(\theta_i, \boldsymbol{f}(\theta))$, where $\pi \in \Delta\Theta$ are given *prior beliefs* of agents about their mutual intentions (a probability distribution over $\{w, x\}^n$). In the tradition of Bayesian games [153, 168], these beliefs are common knowledge of all agents.

A vector $\boldsymbol{f}$ is a *Bayesian Nash equilibrium* [153] in the game $\Gamma(\to, \pi)$ if $\forall i \in N$, $\forall \xi_i \in F_i : (EU_i(\boldsymbol{f}) \geq EU_i(\xi_i, \boldsymbol{f}_{N \backslash \{i\}}))$. Because the payoff functions are supermodular, it is possible to show that such an equilibrium always exist.

Minimal sufficient networks. If all agents have rather "optimistic" beliefs, they will participate in the collective action under any structure of the communication network. As noted in [47], of major concern are the properties of communication networks under which the whole[4] group of agents prefers participation regardless of the prior beliefs. Such networks were called sufficient. So, for a *sufficient network*, there exists an equilibrium in which all agents decide to participate in the collective action under any prior beliefs (recall this is possible only if all agents from the group are willing to do it).

Minimal sufficient networks are remarkable for the absence of redundant communication links. Formally they are defined in the following way. A network $\to$ is minimal sufficient if it is sufficient and, for any other sufficient network such that $\to' \subset \to$, we have the relationship $\to' = \to$. So it does not contain smaller sufficient networks.

In fact, such minimal sufficient networks represent hierarchies of cliques. A *clique* is a subset of agents in which each agent directly informs each other agent. Thus, a clique of the network $\to$ is a set $M_k \subseteq P$ such that $\forall i, j \in M_k$: $i \to j$. Any minimal sufficient network partitions the agents in cliques, and there exists a communication link between cliques in one direction only. A minimal sufficient network arranges in a social network hierarchy of roles defined by cliques and the relation $\to^*$, namely, *leading adopters* and *followers*.

Note that the problem of a minimal structure of communications among the agents that guarantees desired properties of a social network (as a rule, spanning tree in a communication digraph, see [3, 213]) arises for many influence models of social networks—imitation, coordination, Markovian models, etc.

[4]Although, there exist other equilibria in which only some agents participate in the collective action and this is beneficial for them.

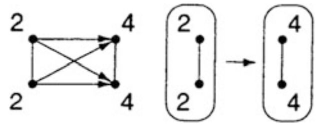


Fig. 1.3 Minimal sufficient network and associated hierarchy of roles

Game of thresholds, local common knowledge. Using several examples, the paper [47] studied a special case $\Gamma_{e_1 \ldots e_n}$ of the game-theoretic model as follows. Agent i decides to participate in a collective action if at least e_i agents do so (threshold). So the payoff function of agent i has the form

$$u_i(w, a) = \begin{cases} 1 \text{ if } a_i = r \text{ and } \#\{j \in N : a_j = r\} \geq e_i, \\ -1 \text{ if } a_i = r \text{ and } \#\{j \in N : a_j = r\} < e_i, \\ 0 \text{ if } a_i = s. \end{cases}$$

(Here #D denotes the cardinality of a set D.)

Each agent has a specific threshold—an integer value between 1 and n + 1. If the threshold is 1, then the agent surely participates; 2, surely participates if his/her neighbor does; and so on. Agent i knows his/her own threshold and also the thresholds of his/her neighbors only. Besides, the authors [47] actually hypothesized (without explicit statement) that each agent knows the mutual awareness of his/her neighbors.

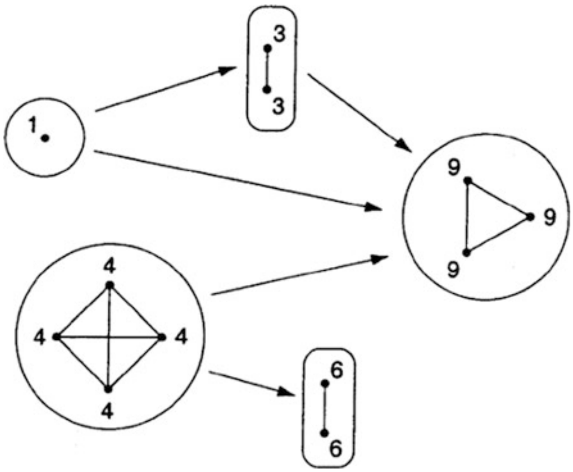


Fig. 1.4 Hierarchy of roles in game $\Gamma_{1,3,3,4,4,4,4,6,6,9,9,9}$

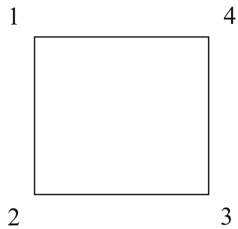


Fig. 1.5 Network of game $\Gamma_{3,3,3,3}$

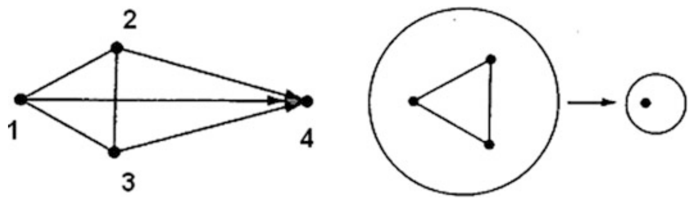


Fig. 1.6 Minimal sufficient network of game $\Gamma_{3,3,3,3}$ and its hierarchy of roles

Example 1.3 The game $\Gamma_{2,2,4,4}$ (Fig. 1.3).

The agents with threshold 2 form the clique of leading adopters while the ones with threshold 4 the clique of followers. •

Example 1.4. The game $\Gamma_{1,3,3,4,4,4,4,6,6,9,9,9}$ (Fig. 1.4).

In this example, we have two leading cliques. Note that the agents from the clique with value 9 should know the threshold of the agent from the clique with value 1: they participate only if the agents from clique 3 do.

Interestingly, here the cliques are homogeneous in the sense that the agents have the same thresholds, and the agents with greater threshold occupy the lower hierarchical level. •

Example 1.5 The game $\Gamma_{3,3,3,3}$ and higher-order beliefs. Consider the game $\Gamma_{3,3,3,3}$ ("square box", see Fig. 1.5) in which all agents are willing to participate (the corner values indicate the indexes of agents, not their thresholds):

Here agent 1 knows that agents 2 and 4 have the same threshold 3. However, he/she is not informed about the threshold of agent 3 and also knows that agent 2 is not informed about the threshold of agent 4. Due to this uncertainty, agent 1 will not participate in the collective action, even despite the existing potentiality (his/her threshold is 3 and the number of other agents willing to participate is 3, as he/she knows). This reasoning applies to each agent. There are rather many agents willing to participate and all they know this fact independently; but none of the agents know that other agents are informed about this. So Example 1.5 illustrates that first-order knowledge can be insufficient for decision-making, and knowledge of higher orders is required.

Consider the game $\Gamma_{3,3,3,3}$ with another structure (minimal sufficient network), see Fig. 1.6.

Agent 1 knows that agents 2 and 3 have the same threshold 3. Moreover, he/she knows that they are informed about the thresholds of each other and that they are informed about his/her knowledge about their mutual beliefs, and so on (the local common knowledge is "All agents 1, 2, and 3 have threshold 3"). The same holds for agents 2 and 3. As it turns out, this common knowledge is sufficient for their participation in the collective action because all agents know about the willingness of each other. Agent 4 also knows this and will participate in the collective action.

This example underlines that the network structure must be the common knowledge of all agents. In other words, an important role is played by cliques in which the appearance of locally common knowledge (an awareness structure for a part of agents) seems quite natural. The information about the players' willingness is coming from leading cliques through chains. The agents know the willingness of other agents but not their actions. Minimal sufficient networks are intrinsic structures of the game interpreted as hierarchical social roles. Communication networks facilitate coordination processes in the following way:

(1) by informing each stage about the earlier/preceding stages;
(2) by forming common knowledge for each stage (role). •

Strong and weak links (ties). It is intuitively clear that a network with very many strong links has small cliques by transitivity: "the friends of my friends become my friends." Therefore, in such networks local common knowledge is formed faster. This happens under rather small thresholds (i.e., the chances that a group with strong link becomes a leading clique). However, if the thresholds are large, then local common knowledge in small cliques remains useless and weak links become dominating: they are running fast through the whole society as well as accelerating communication and the spread of knowledge. So necessary prerequisites for a collective action are created.

1.2.4 Social Control and Collective Action. Network Stability

The paper [115] considered the relationship between the mechanisms of social control, the properties of social networks, and *collective actions* performed for the public goods of a whole community of agents. As established therein, the key factors for agents' decision-making under conflict of private and public interests are different types of social control implemented through interpersonal relations in a social network, namely, *behavioral confirmation* (an agent follows social expectations) and *social selective incentives* (an agent receives additional personal goods from other agents).

More specifically, collective decision-making was modeled by a noncooperative game with single interaction of $n > 2$ agents—*the structurally embedded public goods game* [115]. Denote by $N = \{1, \ldots, n\}$ the set of agents; in this model, each agent $i \in N$ chooses between participation ($\sigma_i = 1$) or non-participation ($\sigma_i = 0$) in a collective action. Participation incurs total cost c and also produces additional good α for each network agent (if $c > \alpha$, then participation of sufficiently many agents n^* can be profitable: $\alpha n^* > c$). Let $r_{ij} = 1$ if there exists an undirected edge (connection) between agents i and j and $r_{ij} = 0$ otherwise. The communication graph has no loops and the total number of connections of agent i makes up $r_i = \sum_{j=1}^{n} r_{ij}$. Each agent undergoes the influence of several factors through his/her connections as follows: behavioral incentives from the neighbors with the same decision (an additive incentive b_1 and a proportional incentive b_2) and also social incentives (s from each neighbor). For the sake of simplicity, assume c, $\alpha > 0$ and b_1, b_2, $s \geq 0$. Designate as C the set of participants and by D the set of free riders. Then $r_i = r_{id} + r_{ic}$ for participant i. The payoffs of agent i from participation and nonparticipation have the form

$$\pi_i(\sigma_i = 1) = r_i s + r_{ic} b_1 + \frac{r_{ic}}{r_i} b_2 + \alpha(\sum_{j=1}^{n} \sigma_j + 1)$$

and

$$\pi_i(\sigma_i = 0) = c + r_{id} b_1 + \frac{r_{id}}{r_i} b_2 + \alpha \sum_{j=1}^{n} \sigma_j,$$

where $j \neq i$.

Therefore, a rational agent benefits from participation if

$$r_i s + (r_{ic} - r_{id})(b_1 + \frac{b_2}{r_i}) + \alpha \geq c.$$

Again, for simplicity let $r_i > 0$, $i \in N$. Then, for all agents, participation in the collective action is a Nash equilibrium if the network satisfies the property

$$\min_i \{r_i\} \geq \frac{c - \alpha - b_2}{s + b_1}.$$

Clearly, for any agent nonparticipation is not beneficial if all his/her neighbors decide to participate. In fact, this means that stronger incentives as well as a higher minimal node degree of the network increase the probability of successful collective action.

However, in some cases only a part of agents n^* may participate but this collective action is beneficial (under $\alpha\, n^* > c$) and also forms a Nash equilibrium if nonempty sets C and D satisfy the following inequalities:

$$\forall i \in C: \ r_i s + (r_{ic} - r_{id})(b_1 + \frac{b_2}{r_i}) + \alpha \geq c;$$
$$\forall j \in D: \ r_j s + (r_{jc} - r_{jd})(b_1 + \frac{b_2}{r_j}) + \alpha \leq c.$$

Whenever there are several Nash equilibria, for each agent choose the equilibrium with higher payoff. If the participation and non-participation of all agents are two Nash equilibria and the number of agents exceeds n^*, then the Nash equilibrium with complete participation is Pareto dominating. If complete and partial participation are two Nash equilibria, then the former is dominating the latter if the payoff function possesses the property

$$\forall j \in D: \; n_d\alpha + r_j s + r_{jc}(b_1 + \frac{b_2}{r_j}) > c.$$

In other words, the fewer free riders the Nash equilibrium with partial participation contains, the rarer its counterpart with complete participation is dominating: free riders have higher chances to benefit more.

Also the paper [115] discussed *the possibility to create and break connections in a social network*. A social network with a given strategy vector is *stable* if, for any agent, the break and/or creation of new connections do not improve his/her payoff. Let a be the cost of connection break and f be the cost of connection creation.

A free rider $i \in D$ benefits from structural changes that create y new connections with other free riders and break x old connections with participants if

$$y\, b_1 + b_2 \frac{y\, r_{ic} + x\, r_{id}}{r_i(r_i - x + y)} > x\, a + y f.$$

A participant $j \in C$ benefits from structural changes that create y new connections with other participants and break x old connections with free riders if

$$(y - x)s + y\, b_1 + b_2 \frac{y\, r_{jd} + x\, r_{jc}}{r_j(r_j - x + y)} > x\, a + y f.$$

Assume the creation and break of connections incur zero cost. Then the only stable network is the network in which the sets C and D are fully connected and have no connections with each other.

A stable network equilibrium was defined in [115] as a strategy profile in which, for any agent, any combination of changes in his/her actions and connections does not improve his/her payoff. As proved by the authors, only the equilibria with complete participation or complete non-participation are stable network equilibria (($s > 0$ or $b_1 > 0$ or $b_2 > 0$) and ($f = a = 0$)).

The model under consideration has some restrictions as follows. First, the connections between agents are undirected; second, for a given agent the incentives provided by all his/her neighbors are equivalent; third, only external incentives are described (each agent is assumed to have no internal incentives); and fourth, the agents are rational and fully informed (this hypothesis is often inapplicable to large networks). A promising approach is to limit awareness (e.g., structurally) and/or to consider the agents of bounded rationality [196].

1.3 Models and Properties of Social Networks

Generally speaking, the presented survey of such an intensively developed field of investigations as influence modeling in social networks allows to draw the following conclusion. Influence models of social networks are still on their way towards becoming an independent discipline of research; today they represent a synthetic “alloy” of graph theory, game theory, social psychology, social theory of small groups, theory of Markov chains, *mechanism design*, theory of multiagent systems and other disciplines. Nevertheless, it can be said with confidence that in the coming years models of social networks will form an independent branch of investigations, attracting more and more scientists in applied mathematics, psychology, economics and sociology. Note that this book has left behind numerous studies of specific social networks based on the models considered. Even a brief description of such studies would require a survey of comparable scope.

The results presented in Sects. 1.1 and 1.2 testify that the modern models of social networks (see their classification in Sect. 1.1.1) are well reflecting many properties and effects of real social networks as listed in the Introduction. Tables 1.1 and 1.2 summarize different classes of models (columns) and properties

Table 1.1 Optimization and simulation models of social networks and their properties

Classes of models properties	Threshold models	Independent cascade models	Models of percolation and contagion	Ising models	Cellular automata models	Markovian models
Individual “opinions” (states) of agents	+	+	+	+	+	+
Variable opinions under an influence of other network members	+	+	+	+	+	+
Different significance of opinions (influence, trust) of given agents for other agents	+	+	+	•	+	•
Different degree of agents’ susceptibility to influence	+	–	–	–	+	–
Indirect influence	–	–	–	–	–	–
Opinion leaders	+	•	–	–	–	–
A threshold of sensitivity to opinion variations of a neighborhood	+	–	–	–	+	–
Local groups	–	•	–	–	–	–

(continued)

Table 1.1 (continued)

Classes of models properties	Threshold models	Independent cascade models	Models of percolation and contagion	Ising models	Cellular automata models	Markovian models
Specific social norms	–	–	–	–	–	–
Social correlation factors	•	–	–	–	–	–
External factors of influence	+	–	–	+	+	–
Stages	+	–	–	–	•	–
Avalanche-like effects (cascades)	+	+	•	–	–	–
The influence of structural properties of social networks on opinion dynamics, including connections, clustering, local mediation, and diameter	+	•	–	–	+	–
Active agents	–	–	–	–	–	–
Possible groups and coalitions of agents	–	–	–	–	–	–
Incomplete and/or asymmetric awareness of agents, decision-making under uncertainty	–	–	–	–	–	–
Nontrivial mutual awareness (reflexion) of agents	–	–	–	–	–	–
Game-based interaction of agents	–	–	–	–	–	–
Optimization of informational influence	+	+	–	–	•	–
Informational control in social networks	–	–	–	–	–	–

Table 1.2 Game-theoretic models of social networks and their properties

Classes of models properties	Models of mutual awareness	Models of coordinated collective actions	Models of communication processes	Models of network stability	Models of informational influence and control	Models of informational confrontation
Individual "opinions" (states) of agents	+	+	+	+	+	+
Variable opinions under an influence of other network members	+	+	+	+	+	+
Different significance of opinions (influence, trust) of given agents for other agents	–	•	•	–	+	+
Different degree of agents' susceptibility to influence	–	•	•	+	+	+
Indirect influence	•	–	–	–	+	+
Opinion leaders	–	–	–	–	+	–
A threshold of sensitivity to opinion variations of a neighborhood	–	–	•	–	–	–
Local groups	–	–	+	–	+	•
Specific social norms	–	–	–	–	+	•
Social correlation factors	•	•	•	–	–	–
External factors of influence	–	•	–	•	+	+
Stages	–	–	+	–	•	–
Avalanche-like effects (cascades)	–	–	–	–	•	–
The influence of structural properties of social networks on opinion dynamics, including connections, clustering, local mediation, and diameter	•	+	+	+	•	•
Active agents	•	+	•	•	•	+
Possible groups and coalitions of agents	–	–	–	–	•	•

(continued)

Table 1.2 (continued)

Classes of models properties	Models of mutual awareness	Models of coordinated collective actions	Models of communication processes	Models of network stability	Models of informational influence and control	Models of informational confrontation
Incomplete and/or asymmetric awareness of agents, decision-making under uncertainty	+	+	+	–	•	+
Nontrivial mutual awareness (reflexion) of agents	+	–	•	–	+	•
Game-based interaction of agents	•	+	•	+	•	+
Optimization of informational influence	–	–	–	–	+	+
Informational control in social networks	–	–	–	–	+	+

(rows) of social networks. The notations are as follows: "+" if a corresponding model gives an adequate description for a corresponding property and "•" if takes into account.

The deep analysis of this chapter and also the brief summary in Tables 1.1 and 1.2 indicate that a series of important properties of social networks still have to be examined through an adequate modeling framework.

Chapter 2
Models of Informational Control in Social Networks

In this chapter, we develop and study game-theoretic and optimization models and methods of informational influence and control in social networks. Section 2.1 considers a model of informational influence with focus on the formation and dynamics of agents' opinions in a social network. Opinion dynamics is described by a Markov process while opinions are calculated using an influence graph. We introduce the concepts of communities, groups, and satellites. Based on the resulting influence structure, we prove that the opinions of satellites are defined by the opinions of groups while the opinions within groups are converging (stabilized) to the same value.

In Sects. 2.2–2.6 this model of informational influence is employed to design models of informational control for

- the opinions of social network members (Sects. 2.2, 2.3 and 2.6);
- the reputation of social network members (Sect. 2.4);
- the trust of social network members (Sects. 2.2, 2.4 and 2.5).

As demonstrated below, the stable network state is linear in the control variable. Next, we define the concept of *reputation* and consider models of informational control and confrontation for describing the reputation dynamics of social network members and examining the role of reputation in informational influence. Interestingly, the higher is the reputation of an *active agent* who performs manipulation, the greater are his/her capabilities to influence the resulting opinion of all other agents in a social network. Therefore, as follows from the model, an active agent maximizes his/he reputation by manipulating his/her initial opinions on each issue in order to guarantee a desired resulting opinion of all social network members on the last issue. So, in this setup, informational confrontation (a game of several manipulating agents) is actually reduced to dynamic active expertise with reputation. Also some approaches to model the strategic and informational reflexion of agents are analyzed.

A. G. Chkhartishvili et al., *Social Networks: Models of Information Influence, Control and Confrontation*, Studies in Systems, Decision and Control 189,
https://doi.org/10.1007/978-3-030-05429-8_2

Section 2.7 deals with a model of actions spreading in a social network and an associated influence calculation method. In this model, a basic element is an action performed by an agent (network user), which explains the term "actional model."

Note that the models considered in this chapter well reflect many phenomena occurring in real social networks (see the Preface).

2.1 Markovian Model of Informational Influence

Below we will study the formation and dynamics of opinions in a social network using Markov chains: opinion dynamics will be described by a Markov process while the opinions themselves will be calculated by an influence graph. Generally speaking, this model follows the tradition of social network modeling with Markov chains, see [51, 66, 98] and also [180].

The conclusions established within the framework of this model well match the results of social psychologists (e.g., see [48, 152, 190, 221]). For example, the same opinion is gradually formed in a group of closely connected vertexes.

Direct and indirect informational influence. Denote by $N = \{1, 2, \ldots, n\}$ the set of agents belonging to a social network. Agents are influencing each other in the network, and their influence levels are defined by a direct influence matrix A of dimensions $n \times n$. For each agent i, an element $a_{ij} \geq 0$ specifies his/her degree of trust to agent j. Throughout the book, we will operate the concepts of influence and trust as opposite ones in the following sense. The statements "The degree of trust of agent i to agent j is a_{ij}" and "The influence level of agent j on agent i is a_{ij}" are equivalent.

The degrees of trust over a social network can be described by arrows between different vertexes with appropriate weights. For example, an arrow from agent i to agent j with a weight a_{ij} (see Fig. 2.1) indicates that the degree of trust of agent i to agent j is a_{ij}.

Assume agent i knows only his/her own (ith) row of the matrix A, i.e., whom and how much this agent trusts.

Accept the normalization condition:

$$\forall i \in N\colon \sum_{j=1}^{n} a_{ij} = 1. \tag{2.1}$$

In other words, the "total degree of trust" of any agent is 1. This condition guarantees that the matrix A is stochastic in rows [70]. Note that each agent can trust him/herself, which corresponds to the inequality $a_{ii} > 0$.

If agent i trusts agent j while agent j trusts agent k (see Fig. 2.2), then agent k has *indirect influence* on agent i (yet the latter may know nothing about the former!).

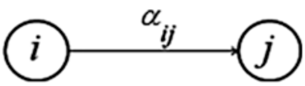


Fig. 2.1 Direct influence (trust)

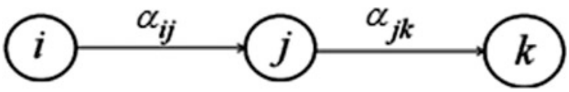


Fig. 2.2 Indirect trust (influence)

This fact motivates us to explore the evolvement of agents' opinions in a social network.

Formation and dynamics of agents' opinions. Assume at some time each agent has a specific opinion on some issue. The opinion of agent i is described by a real value x_i^0, $i \in N$. So the opinions of all agents in the network form a column vector of opinions x^0 of dimension n.

Agents are interacting with each other by exchanging their opinions. As a result, the opinion of each agent varies in accordance with the opinions of other agents he/she actually trusts. Let this variation be linear in the following sense. At a subsequent time, the opinion of agent i is given by the weighted sum of the opinions of other agents he/she trusts:

$$x_i^{\tau} = \sum_j a_{ij} x_j^{\tau-1}, \quad i \in N. \tag{2.2}$$

Here τ denotes time and the weight coefficients are the degrees of trust a_{ij} [94].

As easily seen, in the vector representation the first variation of the opinion is the product of the direct influence matrix and the initial opinion vector, i.e., $x^1 = A\, x^0$. For the subsequent times, we may write by analogy $x^2 = (A)^2 x^0, x^3 = (A)^3 x^0$, and so on.

If this interaction of agents takes place for a sufficiently large period, their opinions are stabilized, converging to the resulting opinion $X = \lim_{\tau \to \infty} x^{\tau}$ (the existence conditions will be discussed below).

The resulting influence matrix is the limit $A^{\infty} = \lim_{\tau \to \infty} (A)^{\tau}$ (the existence conditions will be also discussed below). Then we have the relationship

$$X = A^{\infty} x^0, \tag{2.3}$$

where x^0 denotes the initial opinion vector, A^{∞} is the resulting influence matrix, and X gives the resulting opinion vector.

The structure of indirect trust (influence) is well described by a directed graph in which vertexes correspond to different agents and arrows between them to the degrees of trust. (From a given agent an arrow comes to those agents whom he/she trusts; if the degree of trust is zero, no arrow.)

Example 2.1 An example that shows the transformation of direct influences into the resulting ones can be observed in Fig. 2.3.

In accordance with Fig. 2.3b, the resulting influence of network agents is concentrated in two agents, indexes 3 and 6. These two agents actually determine the opinion of the whole social network. •

We will need several concepts to describe the structure of the resulting trust (influence) in the general case as follows.

Groups and communities. *A community* is a set of agents undergoing no influence from the agents outside it. Formally, a community is a subset $S \subset N$ such that $\forall i \in S$, $\forall j \in N \backslash S$: $\alpha_{ij} = 0$. Denote by $\aleph$ the set of all such subsets.

A group is a community of interacting agents in which each agent influences or undergoes influence from each other agent in it, directly or indirectly. Formally, a group is a minimal community in which it is impossible to extract another community, i.e., a set $Q \in \aleph$ such that $\neg\exists S \in \aleph (S \subset Q)$.

A satellite is an agent who undergoes the influence of agents from other groups but does not influence any of them (any agent from one of these groups). This is an agent not belonging to any group.

Therefore, each agent either belongs to a single group or is a satellite. At the same time, an agent may belong to several "nested" communities.

Figure 2.4 illustrates a group, community and satellites for the social network of Example 2.1. Here we have a unique group of agents 3 and 6; the other agents represent satellites.

Structure of resulting influences. We will describe the structure of resulting influences using well-known results of finite Markov chains (e.g., see [122]). To this end, establish the following correspondences between the currently used concepts and the concepts of Markov chains:

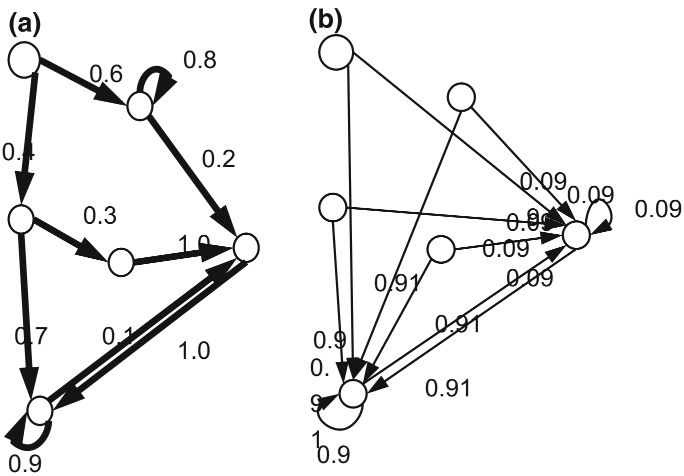
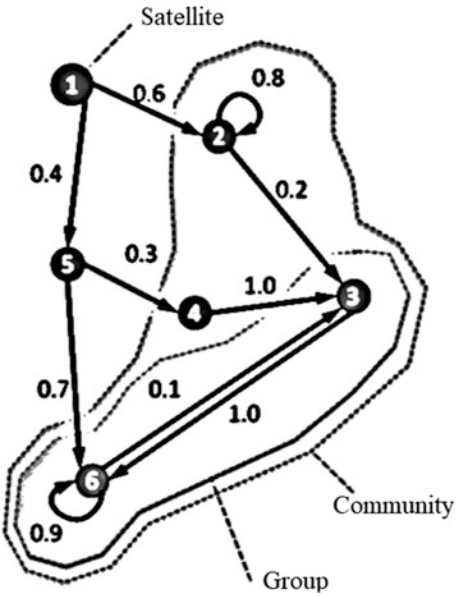


Fig. 2.3 Transformation of direct influence (**a**) into resulting influence (**b**)

Fig. 2.4 Community, group and satellite for social network of Example 2.1



- an agent—a state of a Markov chain;
- the degree of trust of an agent—the probability of transition from one state to another in a Markov chain (transition rate);
- the direct trust matrix—the transition rate matrix;
- indirect trust—reachability;
- a group—an irreducible class of essential states;
- a satellite—an inessential state.

Further exposition relies on Condition 1 unless otherwise stated: in each group there is at least one agent $i \in N$ such that $\alpha_{ii} > 0$. That is, in each group at least a single agent trusts him/herself with some degree.

In this case, each group corresponds to an irreducible aperiodic class in theory of Markov chains. And the following results are immediate from the facts established for Markov chains.

Proposition 2.1 *There exists the resulting influence matrix—the limit* $A^{\infty} = \lim_{\tau\to\infty}(A)^{\tau}$.

Proposition 2.2 *The opinions of all agents are stabilized, converging to the limit* $X = \lim_{\tau\to\infty} x^{\tau}$.

Proposition 2.3 *The resulting influence of any satellite on any agent is* 0. *Particularly, the initial opinions of satellites do not affect the resulting opinions of any agents.*

Proposition 2.4 *The rows of the resulting influence matrix that correspond to the members of one group coincide. Particularly, the resulting opinions of these agents coincide, i.e., each group has a common opinion* (collective opinion).

Note that Proposition 2.4 agrees with the observations of social psychologists: under informational influence, the members of a group reach *consensus*.

Consequently, the structure of resulting influences has the following form, see Fig. 2.5. There are several groups in each of which the resulting opinions of agents coincide (consensus is reached) independently of the initial opinions of other agents not belonging to a given group. The other agents are satellites and their resulting opinions are completely defined by the opinion of one or several groups.

As mentioned earlier, each agent from a group is *an essential state* in the terminology of finite Markov chains[1] [70, 105, 122]. A well-known fact of this theory is that the transition matrix A of a Markov process with several irreducible classes of essential states (here—groups) can be written as

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & A_k & 0 \\ Q_1 & \cdots & Q_k & R \end{pmatrix},$$

where A_l gives the transition matrix within group l (*irreducible* stochastic matrix); k is the number of irreducible classes; Q_l is a matrix that describes the influence of group l on the satellites; the total influence of group l on satellite j is the sum of indexes in the corresponding row of the matrix Q_l.

If a finite Markov process reaches some essential state from a class A_l, then only the essential states from this class are possible at subsequent steps. In addition, the process can return to this essential state only after a finite number of steps (which does not exceed the number of states in this group). The minimal number of steps in which the process returns to an essential state after leaving it is called *the state period*. In the social network model under consideration, this is the length of the minimal cycle in the graph defined by the matrix A_l that passes through the corresponding agent (essential state).

The greatest common divisor for the periods of all essential states from a given class is called its *cyclicity* d_l [122]. As a matter of fact, cyclicity plays a crucial role: the opinions within a separate group l are converging if and only if the mutual influence matrix for the agents from this group is *acyclic* (or *primitive* in the Kolmogorov sense [70]), i.e., $d_l = 1$.

It was demonstrated in [94] that the opinions are converging within a separate group if it contains at least one agent trusting him/herself with some degree. Really, the mutual influence matrix of this group is acyclic because the minimal cycle for the agent has length 1. The matrix A is called *simple* if all its irreducible classes are

[1]The matrix analysis of the structure of resulting influences was performed by N.A. Korgin, Dr. Sci. (Eng.).

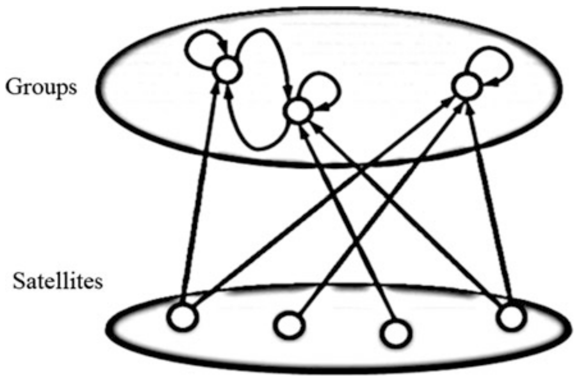


Fig. 2.5 Structure of resulting influence graph

acyclic. A simple matrix A is called *regular* [70, 122] if it contains a unique irreducible class.

For the sake of simplicity, we will assume sometimes (with special mention) that all elements of a stochastic direct influence matrix A are strictly positive. A sufficient condition of regularity is that all members of a social network form a single group. Note that, even under such a strong assumption, there is no guarantee that the opinions of all agents converge to the same value (consensus) in finite time.

For a regular matrix A, each row of the matrix A^{∞} represents the same probabilistic positive vector $\alpha = (\alpha_1, \ldots, \alpha_n)$: $\sum_{i=1}^{n} \alpha_i = 1, \quad \alpha_i > 0, \quad \forall i \in \{1, \ldots, n\}$. Moreover, this vector satisfies the matrix equation $\alpha\, A = \alpha$, which has a unique solution since the matrix A is regular [70, 122]. The vector α is called the *final* or *limit* distribution of a regular Markov chain [122].

Given an initial opinion vector x^0, each agent has the resulting opinion $\alpha\, x^0$. Therefore, within the framework of the current model, the value α_i can be treated as *the influence level of agent i*: it determines how much the resulting opinion reflects the initial one. Also an obvious and interesting fact is that $\forall \tau = 1, 2, \ldots$: $\alpha x^0 = \alpha x^{\tau}$, where $x^{\tau} = (A)^{\tau} x^0$.

So, for any $a \in \Re^1$, we may define *the domain of attraction*—the set of initial opinions from which this value can be reached as the group consensus $X = a \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$:

$$\mathrm{X}(a) \subseteq \Re^{n-1} = \left\{ x^0 \in \Re^n \colon \alpha x^0 = a \right\}.$$

Moreover, $\forall a, b \in \Re^1$, $a \neq b$: $\mathrm{X}(a) \cap \mathrm{X}(b) = \varnothing$ (from any initial opinion vector, it is possible to reach only a single vector of resulting opinions). This means that for different resulting opinions the reachability spaces are parallel. Geometrical interpretations of this statement have close connection with the condition figuring in Proposition 2.6), see Sect. 2.2.

As a digression note that the problem to find the relative influence of agents in a social network (analysis of the equations $\alpha A = \alpha$) is very similar to *the PageRank problem*, see the surveys in [130, 131, 213].

This leads to the question about the awareness of agents in a current situation. Does an agent know that he/she is a member of a group or a satellite? A reasonable approach proceeds from the hypothesis that, at each time, each agent knows his/her own opinion, the opinion of those agents he/she trusts as well as his/her degree of trust to these agents (direct trust). If an agent knows that his/her resulting opinion is different from the opinion of those he/she trusts, then this agent is a satellite and also knows this fact. At the same time, if the resulting opinions of an agent and those he/she trusts coincide, then this agent can be a member of a group or a satellite.

Formation and dynamics of agents' opinions: some examples. In this paragraph, we will consider several typical examples illustrating the formation of agents' opinions.

Our analysis starts with the "limiting" cases.

Example 2.2 Agent $i \in N$ trusts him/herself only, i.e., $\forall j \neq i$: $\alpha_{ij} = 0$ and $\alpha_{ii} = 1$. The opinions of such an agent remain invariable with the course of time: $x_i^\tau = x_i^0$, $\tau = 0, 1, \ldots$. •

Example 2.3 To some degree an agent trusts all other agents, who have the same opinion and trust nobody except themselves. Then the opinion of this agent is converging to the opinion of the other agents, which remains invariable. •

Example 2.4 Each of two agents absolutely trusts the opponent ($\alpha_{12} = \alpha_{21} = 1$). Then Condition 1 above fails, and the opinions of these agents are fluctuating with period 2. •

Example 2.5 For two agents, the situation is symmetric: $\alpha_{11} = \alpha_{22} < 1$. The initial opinions of agents 1 and 2 are 0 and 1, respectively. Then both agents have the same resulting opinion 0.5, and the maximal rate of convergence takes place under the condition $\alpha_{11} + \alpha_{22} = 1$. •

Example 2.6 A social network is a complete graph in which all agents have the same degree of trust to themselves and others. Then the resulting opinion of all agents is the arithmetic mean of their initial opinions. •

Example 2.7 A social network is a linear chain of agents.

A. Agent 9 trusts him/herself only; each of the other agents trusts him/herself and the succeeding agent, $\alpha_{i,i+1} = 0.5$. Then a damping wave of opinions is running over the chain, i.e., the opinion of agent i at a time τ has the form $x_i^\tau = 0.5\,x_i^{\tau-1} + 0.5\,x_{i+1}^{\tau-1}$. The resulting opinions are the same,[2] see Fig. 2.6

[2]The networks and opinion dynamics in this and some other examples below were obtained by simulation modeling.

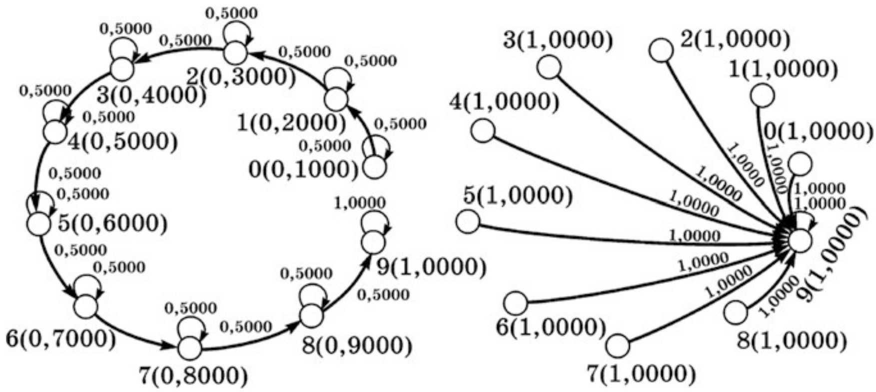


Fig. 2.6 Illustration for Example 2.7, case A

(in this example as well as in the subsequent examples of Sect. 2.1, agents are numbered starting from 0).

B. Each agent trusts him/herself, $\alpha_{i,i} = 0.5$. For agent 0, $\alpha_{0,1} = 0.5$; for agent $n-1$, $\alpha_{n-1,n-2} = 0.5$; for all other agents, $\alpha_{i,i-1} = 0.25$ and $\alpha_{i,i+1} = 0.25$. Then the resulting opinions coincide and are given by $X = \frac{1}{n-1}\left(0.5\,x_0^0 + 0.5\,x_{n-1}^0 + \sum_{i=1}^{n-2} x_i^0\right)$. (Figure 2.7 shows the initial and resulting opinions as well as the degrees of trust for a network of 10 agents.) •

Example 2.8 A social network is a ring.

A. Each agent trusts him/herself, $\alpha_{i,i} = 0.5$, and also the next agent in the ring, $\alpha_{i,i+1} = 0.5$. Then the resulting opinions coincide and are given by $X = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$ (see Fig. 2.8).
B. For each agent i, $\alpha_{i,i} = 0.5$, $\alpha_{i,i+1} = 0.25$, and $\alpha_{i,i-1} = 0.25$. Then the resulting opinions coincide and are given by $X = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$ (see Fig. 2.9). •

Example 2.9 A social network is a star.

A. All agents trust the center, $\alpha_{i,0} = 0.5$, and also themselves, $\alpha_{i,i} = 0.5$. The center of this star (agent 0) trusts only him/herself, $\alpha_{0,0} = 1.0$. Then the resulting opinions of the agents are the initial opinion of the center, $X = x_0^0$ (see Fig. 2.10).
B. The center trusts him/herself, $a_{0,0} = 0.5$, and also the periphery agents, $\alpha_{0,i} = 0.5/(n-1)$. In turn, the latter trust themselves, $\alpha_{i,i} = 0.5$, and also the center, $\alpha_{i,0} = 0.5$. Then the resulting opinions coincide and are given by $X = 0.5x_0^0 + \frac{0.5}{n-1}\sum_{i=1}^{n-1} x_i^0$ (see Fig. 2.11). •

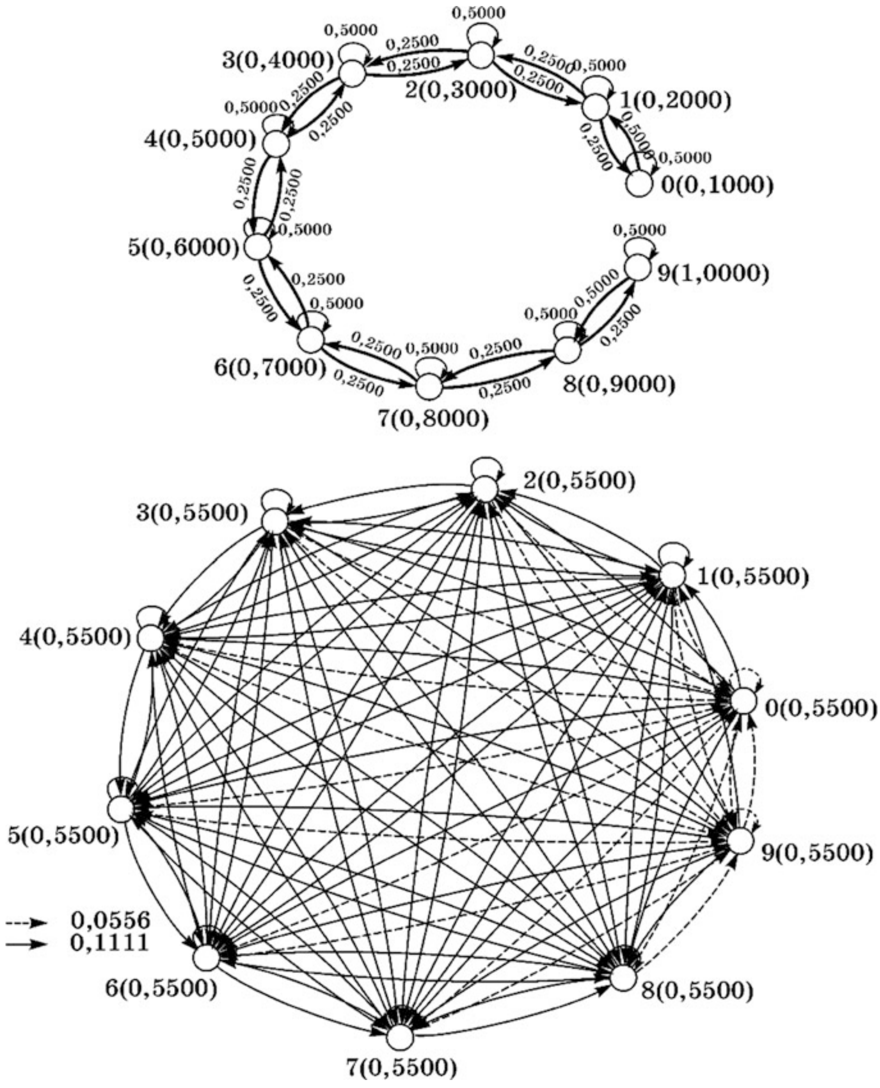


Fig. 2.7 Illustration for Example 2.7, case B

Example 2.10 A social network is a complete graph. Each agent trusts him/herself, $\alpha_{i,i} = 0.5$, and also other agents, $\alpha_{i,j} = 0.5/(n - 1)$. Then the resulting opinions of all agents coincide and are given by $X = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$ (see Fig. 2.12). •

Example 2.11 A social network consists of two complete graphs (of n and m agents, respectively). In the first graph, each agent trusts him/herself, $\alpha_{i,i} = 0.5$,

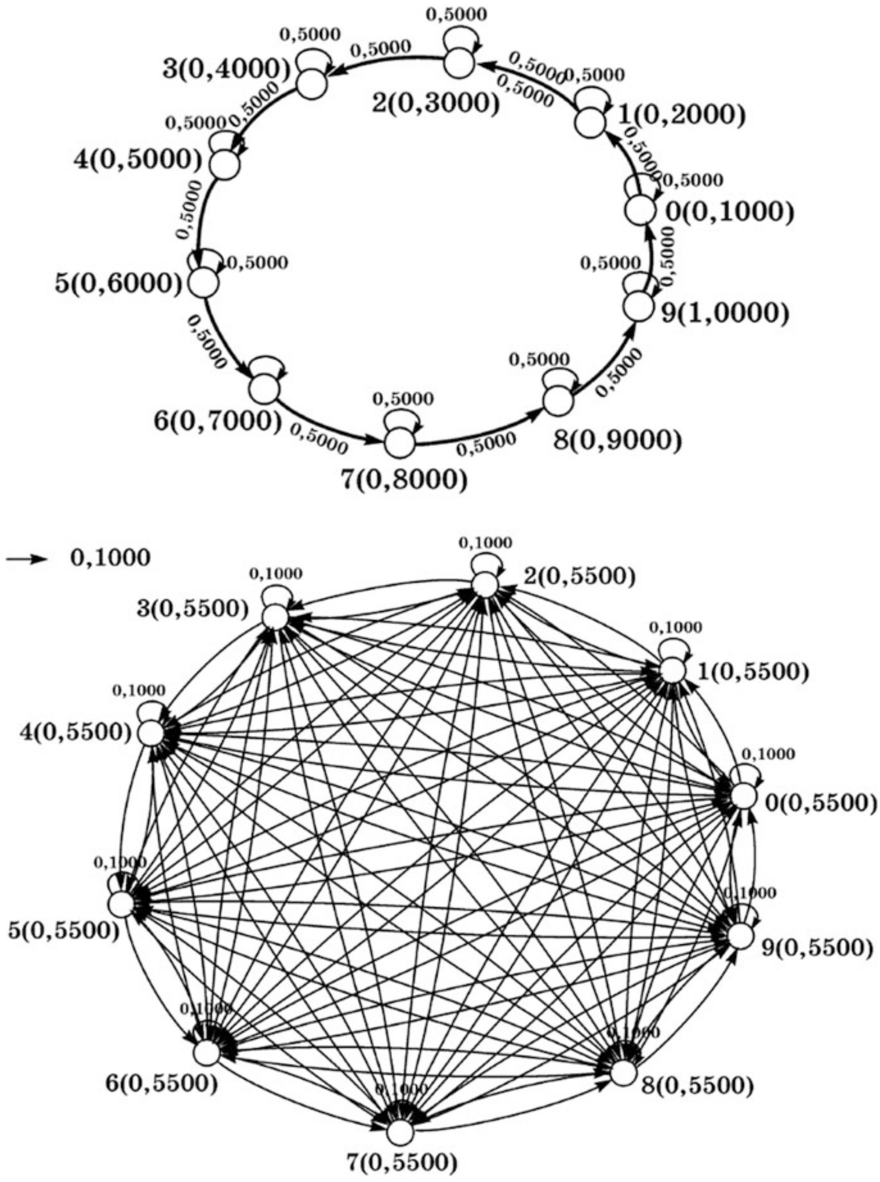


Fig. 2.8 Illustration for Example 2.8, case A

and $\forall j \neq i$: $\alpha_{ij} = \frac{0.5}{n-1}$. The second graph is connected with the first through an incoming arc. Then the resulting opinions of all agents coincide and are given by $X = \frac{1}{n}\sum_{i=0}^{n-1} x_i^0$ (see Fig. 2.13). •

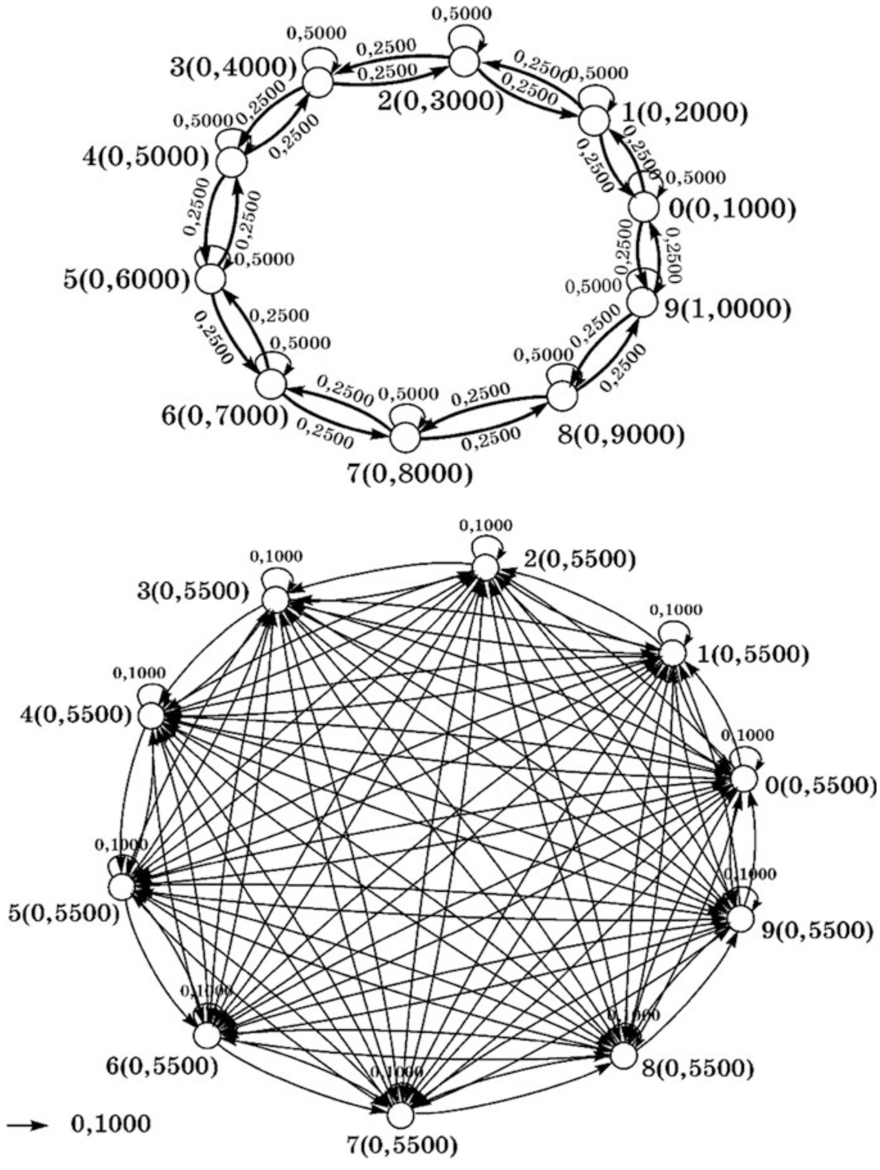


Fig. 2.9 Illustration for Example 2.8, case B

Example 2.12 A social network is a regular tree.

A. The agent at the root vertex (agent 0) trusts him/herself only while the other agents trust themselves and the parent agent, $\alpha_{i,i} = 0.5$. Then the resulting opinions of all agents coincide and are given by $X = x_0^0$ (see Fig. 2.14).

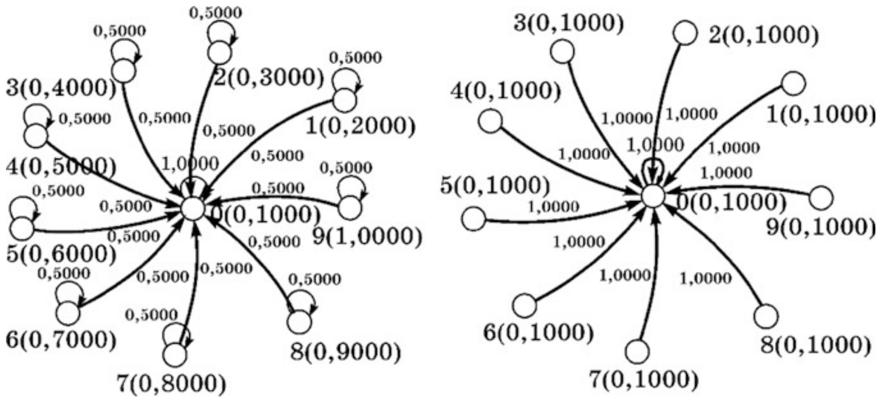


Fig. 2.10 Illustration for Example 2.9, case A

B. Each vertex in the tree has m descendants; each agent trusts him/herself, $\alpha_{i,i} = 0.5$, and also the parent agent and the descendants. Denote by N_l the set of leaves, by N_{int} the set of intermediate vertexes, and by r the root, i.e., $N = N_{\text{int}} \cup N_l \cup \{r\}$. Then, as shown in Fig. 2.15,

$$X = \frac{mx_r^0 + (m+1)\sum_{i \in N_{\text{int}}} x_i^0 + \sum_{i \in N_l} x_i^0}{m + (m+1)|N_{\text{int}}| + |N_l|}$$

•

Example 2.13 Each of three agents trusts him/herself and others with some degree. The agents have different initial opinions. Then the opinions are converging to the same resulting opinion for all agents.

This conclusion can be illustrated with an experiment conducted by well-known Turkish–American psychologist Sherif [152].

As a participant in one of Sherif's experiments, you might have found yourself seated in a dark room. Fifteen feet in front of you a pinpoint of light appears. At first, nothing happens. Then for a few seconds it moves erratically and finally disappears. Now you must guess how far it moved. The dark room gives you no way to judge distance, so you offer an uncertain "six inches." The experimenter repeats the procedure. This time you say, "Ten inches." With further repetitions, your estimates continue to average about eight inches. The next day you return to the darkened room, joined by two other participants who had the same experience the day before. When the light goes off for the first time, the other two people offer their best guesses from the day before. "One inch," says one. "Two inches," says the other. A bit taken aback, you nevertheless say, "Six inches." With repetitions of this group experience, both on this day and for the next two days, will your

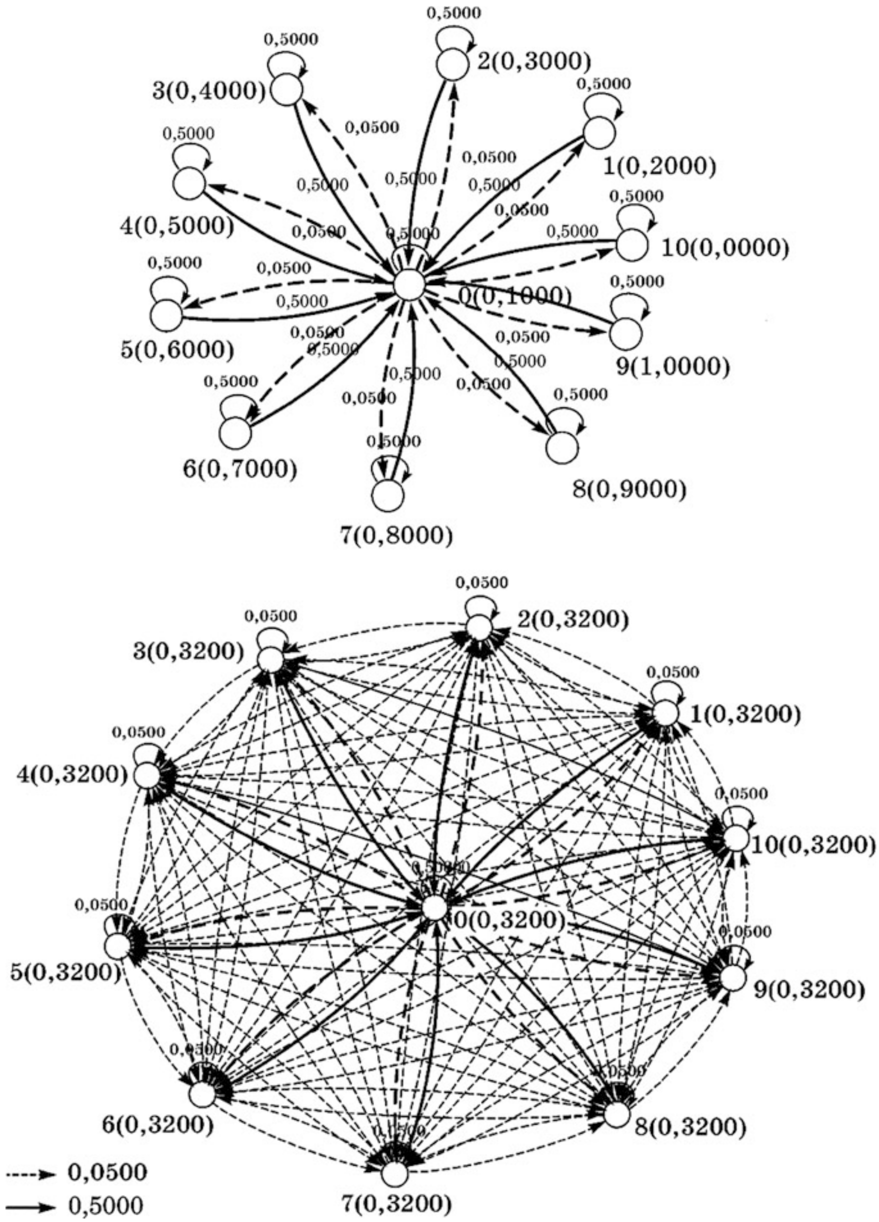


Fig. 2.11 Illustration for Example 2.9, case B

responses change? The Columbia University men whom Sherif tested changed their estimates markedly. A group norm typically emerged. The norm was false. Why? The light never moved! •

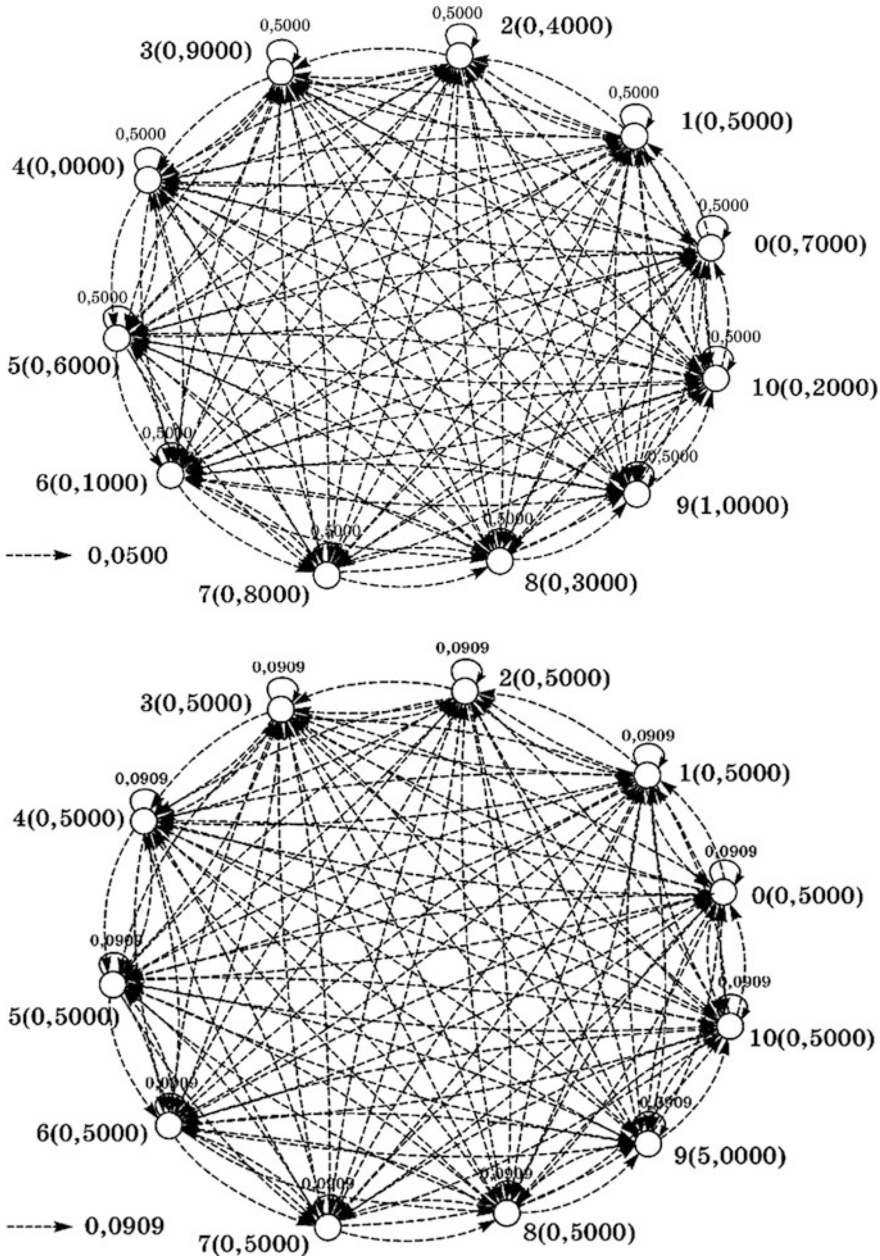


Fig. 2.12 Illustration for Example 2.10

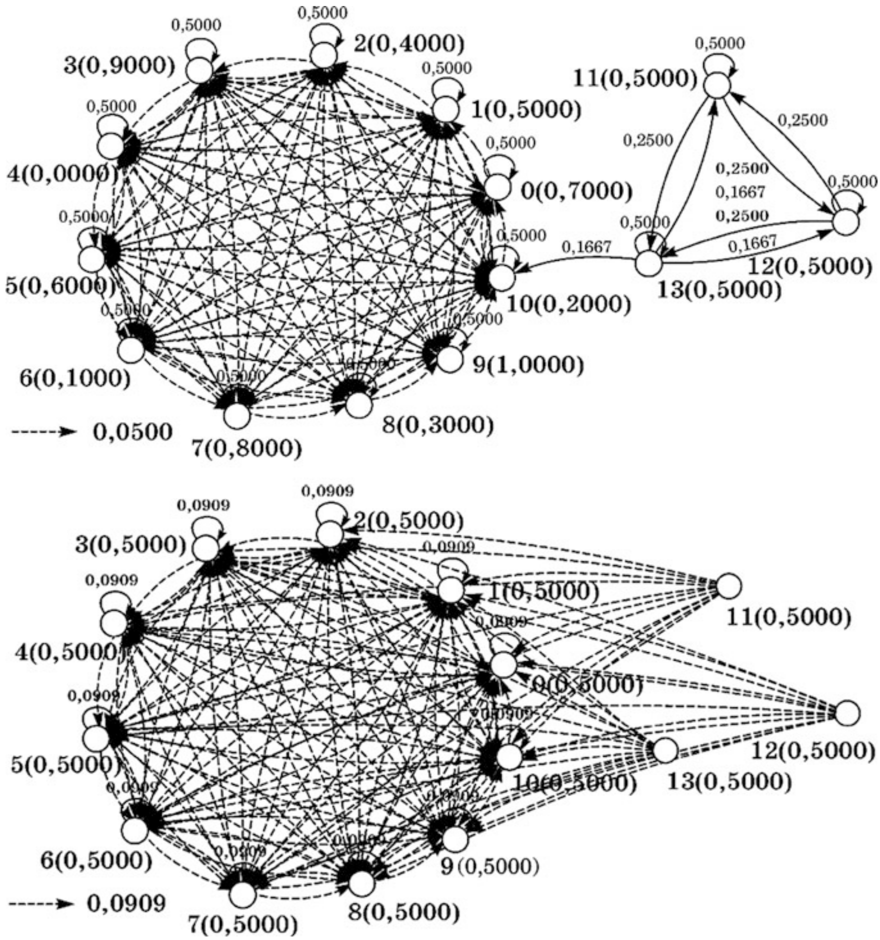


Fig. 2.13 Illustration for Example 2.11

Example 2.14 Five of six agents trust only themselves while agent 6 him/herself and also all other agents to some degree. The initial opinions of agents 1–5 are 0; of agent 6, 1. Then with the course of time the opinion of agent 6 will converge to that of the other agents (0), which will remain invariable.

This conclusion is well illustrated by the Asch experiment [152].

You are seated sixth in a row of seven people. The experimenter explains that you will be taking part in a study of perceptual judgments, and then asks you to say which of the three lines matches the standard line. You can easily see that it's line 2. So it's no surprise when the five people responding before you all say, “Line 2.” The next comparison proves as easy, and you settle in for what seems a simple test. But the third trial startles you. Although the correct answer seems just as clear-cut, the first person gives a wrong answer. When the second person gives the same wrong answer, you sit up in your chair and stare at the cards. The third person

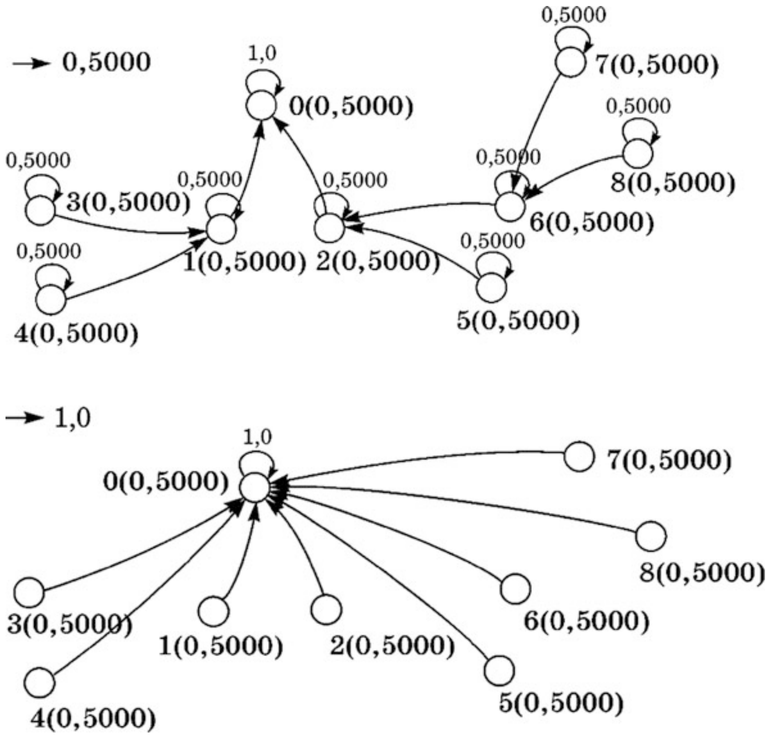


Fig. 2.14 Illustration for Example 2.12, case A

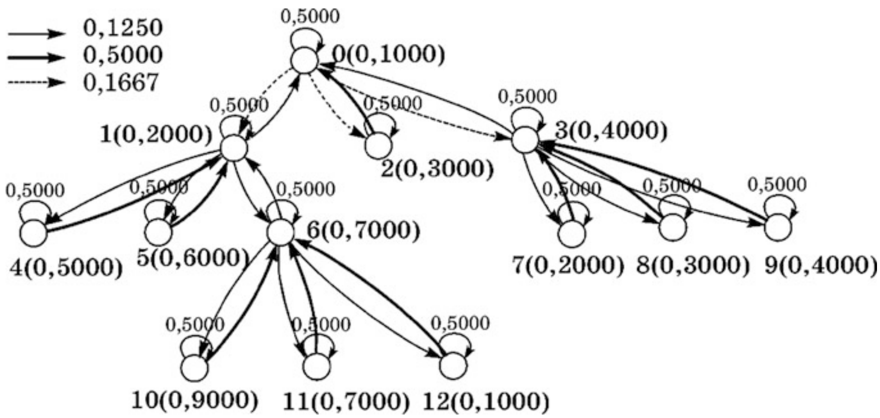


Fig. 2.15 Illustration for Example 2.12, case B

agrees with the first two. Your jaw drops; you start to perspire. "What is this?" you ask yourself. "Are they blind? Or am I?" The fourth and fifth people agree with the others. Then the experimenter looks at you. Now you are experiencing an epistemological dilemma: “What is true? Is it what my peers tell me or what my eyes tell me?” Dozens of college students experienced that conflict in Asch’s experiments. Those in a control condition who answered alone were correct more than 99% of the time. Asch wondered: If several others (confederates coached by the experimenter) gave identical wrong answers, would people declare what they would otherwise have denied? Although some people never conformed, three-quarters did so at least once. All told, 37% of the responses were conforming (or should we say “trusting of others”). •

Therefore, these two examples well match the observations the social psychologists.

Let us draw some intermediate conclusions. It should be acknowledged that the Markovian model (basic model) represents perhaps the simplest model of influence in social networks with reputation of agents. Possible extensions of this model are obvious: reject the hypothesis about graph’s completeness; define a more complicated relationship between trust/influence and reputation; consider the opinions of agents with weights depending on the deviations from some average opinion; take into account the mutual assessments of agents; and so on. The basic model will be employed below even despite inherent simplicity: it yields a series of analytic solutions for informational control and confrontation in social networks.

Note that *multinetworks* form another promising direction of informational influence modeling in social networks, which proceeds from the following idea. Each subject is a member of several (real and/or virtual) social networks simultaneously. For example, online social network *Odnoklassniki* popular in the RuNet contains the subnetworks of graduates of particular colleges or universities, the subnetworks of people with particular hobby, and so on. Actually everybody has certain social roles in different social networks (job, family, friends, etc.). A formal description of a multinetwork is a set of subgraphs on the same set of vertexes. How are these networks intersecting in the mind of a person? This question has not been given an adequate answer so far. For associating different networks with intersecting (or even coinciding) sets of participants, we may hypothesize that each agent needs some time for participation, which is limited. So the multinetwork effects can be described with a certain time allocation model for an agent.

2.2 Informational Control and Opinions of Network Members[3]

Static model of informational control. With the basic equation that connects the initial and resulting opinions of all agents (see Formula (2.2) of Sect. 2.1), we may pose and solve a control problem—influence the social network agents in order to form necessary opinions. For preserving additivity, assume a control subject

[3]This section was written jointly with I.N. Barabanov, Cand. Sci. (Phys.-Math.).

(Principal) knows the influence (trust) matrix. An informational influence (control) is that the Principal modifies the initial opinion vector x^0 by "adding" a control vector $u \in \Re^n$. In fact, control means that the opinion of agent i is changed from x_i to $x_i + u_i$, $i \in N$.

Suppose $u_i \in U_i, i \in N$ (this constraint has clear practical interpretations). Denote $U = \prod_{i \in N} U_i$.

Then the resulting opinions are determined by the equation

$$X = A^{\infty}\left(x^0 + u\right), \tag{2.4}$$

or, in the component-wise form,

$$X_{ui} = \sum_{j \in N} A_{ij}^{\infty}(x_j^0 + u_j) = \sum_{j \in N} A_{ij}^{\infty} x_j^0 + \sum_{j \in N} A_{ij}^{\infty} u_j, \quad i \in N.$$

So the resulting opinion of agent i is the sum of his/her "undisturbed" resulting opinion $\sum_{j \in N} A_{ij}^{\infty} x_j^0$ and the changes $\sum_{j \in N} A_{ij}^{\infty} u_j$ caused by control. Note that due to (2.4) the "stable" state of a social network is linear in the control vector [94].

Let the Principal's goal function $\Phi(X, u)$—*the control efficiency criterion*—depend on the resulting opinions of the agents and also on the control vector. Then the control problem is to choose an admissible control vector that maximizes the efficiency criterion

$$\Phi(A^{\infty}\left(x^0 + u\right), u) \rightarrow \max_{u \in U}.$$

In accordance with Propositions 2.1–2.4, it makes no sense to influence the opinions of satellites. So, for a given trust matrix, we can tell which agents should be subjected to informational control.

Following the tradition of organizational systems control [165], we can separate out two additive components in the Principal's goal function: $\Phi(X, u) = H(X) - c(u)$, where $H(\cdot)$ is the Principal's payoff (income) depending on the resulting opinions of the agents[4] and $c(\cdot)$ is the cost of control (in some models, $c = c(x^0, u)$ would be more appropriate).

Example 2.15 Let the opinions of different agents reflect their degree of "assurance" in what the Principal would like to persuade them (support a given candidate in elections, purchase a certain good, make some decision, etc.). Then possible income functions $H(\cdot)$ are the following:

---

[4]In reflexive game theory [168], the actions of different subjects are often determined by their awareness (available information). Therefore, assuming this dependence known, we can pass from the Principal's preferences that depend on agents' actions (which seems quite natural) to the Principal's preferences that depend on the awareness (i.e., opinions) of the agents.

(a) $\frac{1}{n}\sum_{i \in N} X_i$, the average opinion of all agents;
(b) $\sum_{i \in N} \lambda_i X_i$, the weighted opinion of all agents with weights $\lambda_i \geq 0$, $\sum_{i \in N} \lambda_i = 1$;
(c) $n_\theta = |\{i \in N | X_i \geq \theta\}|$, the number of agents whose opinions exceed a threshold $\theta \in [0, 1]$ (in threshold voting, the share of such agents);
(d) $\min_{i \in N} X_i$, the worst opinion among all agents and so on, depending on problem statement and interpretations. •

Example 2.16 Let $H(X) = \frac{1}{n}\sum_{i \in N} X_i$ and assume the Principal's cost is uniform and linear in the control vector, i.e., $c(u) = \beta \sum_{i \in N} u_i$ (in essence, β gives the cost of a unit change in any agent's opinion) while the Principal's resources are limited by a value $R \geq 0$:

$$\beta \sum_{i \in N} u_i \leq R. \tag{2.5}$$

In this case, the control problem becomes the following linear programming (LP) problem:

$$\frac{1}{n}\left(\sum_{i \in N}\sum_{j \in N} A_{ij}^{\infty} x_j^0 + \sum_{i \in N}\sum_{j \in N} A_{ij}^{\infty} u_j\right) - \beta \sum_{i \in N} u_i \to \max_{\{u_i \geq 0\},(2.5)}.$$

Designating $F_j = \frac{1}{n}\sum_{i \in N} A_{ij}^{\infty}, j \in N$, we write it as

$$\sum_{j \in N} (F_j - \beta) u_j \to \max_{\{u_i \geq 0\},(2.5)}. \tag{2.6}$$

The solution of (2.6) is obvious—all the resources should be allocated on changing the opinion of an agent with the maximal value F_j. This solution can be interpreted as follows. In the final analysis, the value F_j reflects the average degree of resulting trust to agent j for all other agents. This characteristic will be called the *influence level* of agent j. All the resource should be utilized to control the agent most trusted by other agents.

This property of the solution of problem (2.6) is caused by the fact that, like the linear programming problem, it has only one constraint (2.5). We can consider a more complex situation by assuming $U_i = [0, R_i]$. For sufficiently small values $\{R_i\}$ (e.g., not exceeding the thresholds above which the agents or some safety systems are detecting external influences), this model can be treated as *hidden control* (see some illustrative examples of such control in [168]). Then a corresponding analog of problem (2.6) has the following solution: arrange the agents in the decreasing order of F_i and sequentially allocate the maximal amount R_i to them until constraint (2.5) becomes crucial. The last agent who receives resources may get them in a smaller amount than the maximal possible. •

Finally, we will discuss promising lines of further research on opinion control social networks.

First, the problem considered in Example 2.16 can be generalized in many directions that correspond to certain setups of *media planning* (in particular, an optimal choice of informational events) and informational resource allocation in advertising, marketing, informational warfare, informational safety, and so on [126].

Second, it seems reasonable to study more complex (in particular, nonlinear) dynamic relationships between the opinions of some agents and the influence of other agents and the Principal (see the classification of networks in the Introduction).

Third, the matrix A^{∞} can be employed for calculating the "influence indices" of agents in other ways, different from the influence level defined above, see [5, 63, 120, 189]. Using certain heuristics and exact solutions with these indices, we may determine top-priority agents for control.

Fourth, it is interesting to solve the controllability problem—find a set of reachable states for a system under given control constraints, see below.

Fifth, due to the additive property (2.4), we can state and solve dynamic problems of finding optimal sequences of informational influences.

Sixth, there are obvious practical interpretations for the inverse problem—find a set of controls (or "minimal" constraints imposed on them) under which a system reaches a given state (or a set of such states), i.e., form required opinions of agents.

Finally, the framework of this model can be used for stating and solving **informational safety** problems: find optimal protection against informational influences on the agents in a social network.

Thus, we have considered a model of informational influence in which the Principal is forming the initial opinions of all agents one-time. Of major interest is to study the capabilities of informational control during several steps. Further analysis deals with such a model [11].

Dynamic model of informational control: analysis. To proceed, consider the case of dynamic informational control as follows. Assume the Principal can influence the opinions of a certain subset $M \subseteq N$ of agents (called *the agents of influence*), not only at an initial time but also at subsequent times. Denote $m = |M|$.

Without loss of generality, let agents 1, 2, … m be those of influence. Designate as $u^k = (u_j^k)_{j \in M}$, $k = 0, 1, \ldots$, the control vector at a time k and consider the matrix $B = \begin{pmatrix} 1 & & \\ & \cdots & \\ & & 1 \\ & 0 & \end{pmatrix}$ of dimensions $n \times m$.

Proposition 2.5 *Let all elements of the stochastic direct influence matrix A be strictly positive and the controls be unlimited. Then any value of the resulting opinions of the social network members is reachable as consensus in the presence of at least one (arbitrary) agent of influence.*

This result follows immediately from the fact that, under the strictly positive elements of the matrix A, all rows of the matrix A^{∞} are identical and have no zero elements (as mentioned earlier, the column sums of this matrix describe the influence levels of corresponding agents) [76]. In accordance with Formula (2.4), e.g., for any value $\frac{1}{n}\sum_{i \in N} X_i$, we may easily find an appropriate control vector.

Assuming that at each time (including zero) the control vector is applied before opinions exchange, write the equation of opinion dynamics in the matrix form as follows [cf. (2.4)]:

$$x^{k+1} = A\left[x^k + B\,u^k\right], \quad k = 0, 1, \ldots . \tag{2.7}$$

This is a difference equation describing a *discrete-time linear control system* [96]. Its solution with a given initial condition (an analog of the solution of the Cauchy problem in the continuous-time case) has the representation

$$x^k = A^k x^0 + \sum_{\tau=0}^{k-1} A^{k-\tau} B\,u^{\tau}, \quad k = 1, 2, \ldots . \tag{2.8}$$

For system (2.7), construct *the controllability matrix* $\Phi_0 = [B'\,A\,B' \ldots A^{n-1}B']$, where $B' = A\,B$.

For the time being, assume there are no control constraints, i.e., $U_j = R^1, j \in M$. Then, the reachability of an arbitrary state x^T of the linear system (2.7) in $T\,(T \geq n)$ times comes to the nondegeneracy of the pair of matrices A and AB or, equivalently, to the equality *rank* $\Phi_0 = n$ [96], where *rank* indicates matrix rank. This equality can be verified in each specific case using the well-known results of the theory of discrete control systems.

If the Principal's preferences depend on the resulting opinions of the agents only, then the problem can be simplified via reduction to the static case.

Proposition 2.6a *Let the Principal apply the controls* $u^0, \ldots, u^l$, $l < +\infty$. *As* $t \to +\infty$, *the resulting opinions of all agents do not change if the same controls* (*in absolute value*) *actions were applied at any other finite times.*

Proposition 2.6b *Let the controls be unlimited. Then, for any finite sequence of the control vectors* $u^0, \ldots, u^l, l < +\infty$, *there exists a control vector* v *at the initial* (*zero*) *time that leads to the same resulting opinions of the agents.*

Proposition 2.6c *Let the controls be unlimited and*

$$span(\Phi_0) \subseteq span\left(A^{l+1}B\right),$$

where $span(\cdot)$ *indicates the linear hull of the matrix columns.*

Then, for any finite sequence of the control vectors $u^0, \ldots, u^l$, $l < +\infty$, and the realized state x^{l+1} of the social network, there exists a control vector $\hat{v}$ at the initial (zero) time that leads to the same network state x^{l+1} at the time $l + 1$.

Proof of Propositions 2.6a–2.6b Using (2.4), (2.7) and the equality $A^{\infty} A = A^{\infty}$, we have:

$$\begin{aligned} X &= A^{\infty}\left[\ldots A\left(A\left(A\left(x^{0}+B\,u^{0}\right)+B\,u^{1}\right)+B\,u^{2}\right)+\cdots+B\,u^{l}\right] \\ &= A^{\infty}\left(x^{0}+B\,u^{0}\right)+A^{\infty}\sum_{\tau=1}^{l} Bu^{\tau}. \end{aligned} \tag{2.9}$$

Denoting

$$v=\sum_{\tau=0}^{l} u^{\tau} \tag{2.10}$$

gives $X = A^{\infty}(x^{0} + B\,v)$, which was to be established.

Proof of Proposition 2.6c In accordance with (2.8), we may write $x^{l+1} = A^{l+1}[x^{0}+B\,u^{0}]+\sum_{\tau=1}^{l} A^{l-\tau+1}B\,u^{\tau}$. On the other hand, it is required to determine a vector $\hat{v}$ such that $x^{l+1}=A^{l+1}[x^{0}+B\hat{v}]$. Under the hypotheses of Proposition 2.6c, by the Kronecker–Capelli theorem we can find the vector $\hat{v}$ as the solution of the following system of linear algebraic equations:

$$A^{l+1}B\hat{v}=A^{l+1}B\,u^{0}+\sum_{\tau=1}^{l} A^{1-\tau}B\,u^{\tau}.$$

Corollary 2.6.1 *Let the controls be unlimited and the Principal's preferences (efficiency criterion) depend only on the resulting opinions of the agents and the sum of controls over the agents and times. Then, for any finite sequence of the control vectors, there exists vector* (2.10) *of the initial controls of at least the same efficiency.*

Therefore, under the hypotheses of Corollary 2.6.1, **the time-dependent control gives nothing new in comparison with the static case**. We emphasize that this result can be fruitful for cognitive map models (see [158] for a discussion of control problems on cognitive maps). Therefore, an essential assumption states that the Principal's preferences depend on the agents' opinions at a finite number $T < +\infty$ of the first times of their interaction. This assumption will be accepted for further exposition.

Let us introduce a series of important definitions. *The influence level* of agent j at a time t is the sum $w_j^t=\sum_{i\in N}(A)_{ij}^t$. *The total opinion* of all agents at a time t is the sum $\sum_{i\in N} x_i^t$. Assume the Principal applied controls u^0, …, u^l. *The total control* is the sum $\sum_{\xi=0}^{l}\sum_{j\in M} u_j^{\xi}$.

For convenient calculations, construct the vector $C=\underbrace{\|1\,1\ldots1\|}_{n}$ and write these characteristics in matrix form: $w^t = C\,A^t$ as a row vector of dimension n that consists of the influence levels of all agents; $x_\Sigma^t = C\,x^t$ as the total opinion of all agents at a time t; $u_\Sigma = \sum_{\xi=0}^{l} C\,B\,u^\xi$ as the total control.

Proposition 2.7 *Let the controls be nonnegative*: $u_j^t \geq 0, j \in M, t = 0, 1, \ldots$. *If the Principal tries to reach the maximum total opinion of all agents at the time T with a given total control, then it suffices to apply a single control at the time t^* on a single agent j^* of maximal influence*:

$$(j^*, t^*) \in \underset{j \in M, t \in \{0,\ldots,T-1\}}{\operatorname{Arg\,max}} \; w_j^{T-t}. \tag{2.11}$$

Proof of Proposition 2.7 *The opinion vector at the time T has the form* [*see* (2.8)]:

$$x^T = A^T x^0 + \sum_{t=0}^{T-1} A^{T-t} B u^t, \quad T = 1, 2, \ldots$$

As before, denote the total influence by

$$u_\Sigma = \sum_{t=0}^{T-1} \sum_{j \in M} u_j^t = CB \sum_{t=0}^{T-1} u^t.$$

Let (j^*, t^*) be a pair (agent, time) that maximizes the agent's influence level, i.e.,

$$(j^*, t^*) \in \underset{j \in M, t \in \{0,\ldots,T-1\}}{\operatorname{Arg\,max}} \; w_j^{T-t}.$$

The total opinion of the agents at the time T satisfies the following chain of relationships:

$$\begin{aligned} x_\Sigma^T &= \sum_{i \in N} x_i^T = \sum_{i \in N} (A^T x^0)_i + \sum_{t=0}^{T-1} \sum_{i \in N} (A^{T-t} B u^t)_i = C A^T x^0 + \sum_{t=0}^{T-1} C A^{T-t} B u^t \\ &= w^T x_0 + \sum_{t=0}^{T-1} w^{T-t} B u^t \leq w^T x_0 + \max_{j \in M, t \in \{0,1,\ldots T-1\}} w_j^t \sum_{t=0}^{T-1} C B u^t \\ &= w^T x_0 + w_{j^*}^{t^*} u_\Sigma. \end{aligned}$$

On the other hand, the controls $u_{j^*}^{t^*} = u_\Sigma, \quad u_j^t = 0\,(j \neq j^*, t \neq t^*)$, turn this inequality into equality. The proof of Proposition 2.7 is complete.

A result similar to Proposition 2.7 holds if the Principal's goal function is partially monotonic in the agents' opinions at any time over the planning horizon and the constraints are imposed on the individual controls instead of the total control. In this case, the optimal controls lie on the boundary of the admissible control set and are applied one-time.

The situation gets complicated if the Principal's goal function is not partially monotonic in the agents' actions. Then the dynamic optimal informational control design comes to a certain optimization problem depending on the structure of the Principal's goal function. Such optimization problems can be solved numerically in each particular case. The linear property of the controlled system [see (2.8)] is an essential factor that simplifies the analysis.

The influence levels of agents may have considerable variations with the course of time, as illustrated by the following example.

Example 2.17 A social network consists of three agents with the influence matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1-\alpha & \alpha \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha \in (0, 1)$ is a constant. This network is defined by the directed graph in Fig. 2.16.

The network has the following structure.

(1) Agent 1 absolutely trusts agent 2.
(2) Agent 2 trusts agent 3 with the degree α and also him/herself with the degree $(1 - \alpha)$.
(3) Agent 3 absolutely trusts him/herself.

As easily checked,

$$A^t = \begin{pmatrix} 0 & (1-\alpha)^{t-1} & 1-(1-\alpha)^{t-1} \\ 0 & (1-\alpha)^t & 1-(1-\alpha)^t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A^\infty = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Calculate the influence levels of different agents: $w_1^t = 0; w_2^t = (2-\alpha)(1-\alpha)^{t-1}$; $w_3^t = 3 - (2-\alpha)(1-\alpha)^{t-1}$.

So the influence level of agent 2 is monotonically decreasing from $(2 - \alpha)$ to 0; of agent 3, monotonically increasing from $(1 + \alpha)$ to 3. In particular, this means that the opinion of agent 3 is dominating on infinite planning horizon while the opinions of agents 1 and 2 become insignificant. If the Principal seeks to maximize the total opinion of all agents, he/she should apply informational control to agent 3.

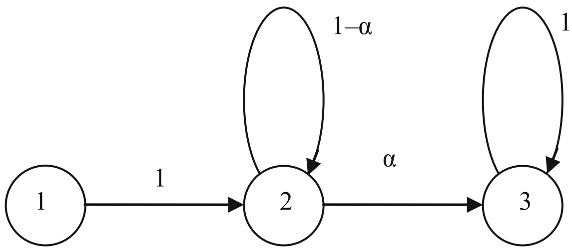


Fig. 2.16 Social network from Example 2.17

However, the situation may change dramatically on finite time horizons. For any $t < \infty$, there exists an interval of sufficiently small values α under which agent 2 is more influential than agent 3 on the whole horizon [0, t]. •

Example 2.18 The Principal is interested in an optimal control for the social network from Example 2.17. It can be calculated by comparing the values w_2^t and w_3^t on a given finite planning horizon [0, t].

If $w_2^t > w_3^t$, then the maximal possible influence should be exerted on agent 2 at the time $\tau = t - 1$; if $w_2^t < w_3^t$, on agent 3 at the time $\tau = 0$ (in the case $w_2^t = w_3^t$, both influences mentioned are optimal). •

Now, we formulate the dynamic problem of optimal informational control design.

Dynamic model of informational control. Design. In the general case, the design problem is formulated as follows. Introduce the following notations: $y = Y(x) \in \Re^k$ as the vector of *observable states* of a social network; $Y: \Re^n \to \Re^k$ as a given function, $k \leq n$; T as a planning horizon; $x^{1,T} = (x^1, x^2, \ldots, x^T)$ as a trajectory of the network states; $y^{1,T} = (y^1, y^2, \ldots, y^T)$ as a the trajectory of the observable network states; $u(y): \Re^k \to U$ as a *control law*; $u^{1,T} = (u(y^1), u(y^2), \ldots, u(y^T))$ as a sequence of controls; finally, $F(y^{1,T}, u^{1,T})$ as a control efficiency criterion.

Let the initial observed state of the social network be known. The general dynamic problem of *optimal positional informational control design* for the discrete system (2.7) is to determine an admissible control law that has the maximum efficiency:

$$F\left(y^{1,T}\left(x^{1,T}\left(u^{1,T}\right)\right), u^{1,T}\right) \to \max_{u(\cdot)}. \tag{2.12}$$

The general dynamic problem of *optimal program informational control design* for the discrete system (2.7) is to determine a sequence of controls that has the maximum efficiency:

$$F\left(y^{1,T}\left(x^{1,T}\left(u^{1,T}\right)\right), u^{1,T}\right) \to \max_{u^{1,T}}. \tag{2.13}$$

Optimal control problems for discrete-time systems were examined by many authors. Some approaches can be found, e.g., in [96].

Consider a series of applications-relevant special cases of problems (2.12) and (2.13). Let a vector y^*, the goal of informational control in the space of observed network states, be fixed.

The problems

$$\left|\left|y^T - y^*\right|\right| \to \min_{u(\cdot)} \tag{2.14}$$

and

$$\left|\left|y^T - y^*\right|\right| \rightarrow \min_{u^{1,T}} \tag{2.15}$$

will be called *the problems of positional and program control for the terminal state of the social network.*

Study problem (2.15), assuming for simplicity that $y = x$ (the states of all system agents are observable). Under the hypotheses of Proposition 2.6c, the minimum in (2.15) is zero and it suffices to apply control only once (see Propositions 2.6a and 2.6b). If the condition $span(\Phi_0) \subseteq span(A^T B)$ fails, then generally the system does not reach the state y^* (here, $x^* = y^*$). So we may endeavor driving the system to some state from the set $A^T x^0 + span(\Phi_0)$ as close to y^* as possible in the Euclidean metric. In this case, the determination of an appropriate control comes to unconstrained minimization of a nonnegative definite quadratic form. Its solution is not unique; by Propositions 2.6a and 2.6b, one of the solutions consists in applying a single-time control to the system.

Problem (2.15) can be reduced to another well-known problem if controls u_i take values from some convex set U, e.g., $|u_i| \leq 1$. Then the determination of a program control driving the system from a given initial state x^0 to a state close to x^* comes to the convex programming problem

$$\left\|A^T x^0 + \sum_{t=1}^{T-1} A^{T-t} B u^t - x^*\right\| \rightarrow \min_{u_i^t \in U},$$

which is solvable by modern methods.

We conclude this section with an example of a positional control problem—the design of a linear controller stabilizing the social network.

Let $y = C_0\, x$, where $C_0 \in \Re^{k \times n}$ is some matrix. Choose a linear control law in the form $u = K\, y$ (if $u = K\, x$, then $C_0 = E_n$ provided that the states of all agents are observable). The closed-loop control system satisfies the equation

$$x^{k+1} = (A + A\, B\, K\, C_0) x^k. \tag{2.16}$$

By the linear property of this system, the stabilization of an arbitrary position x^* is equivalent to the stabilization of the trivial equilibrium of the closed-loop system (2.16). For a stabilizing control, the spectrum of the closed-loop system matrix $A + A\, B\, K\, C_0$ lies inside the unit circle centered at the origin on the complex plane. Note that a corresponding matrix K always exists if $C_0 = E_n$ and the pair A, AB is nondegenerate.

In the general case, the well-known design methods of linear stabilizing controllers for linear discrete-time systems [96] can be used for stabilization of the social networks.

The main result of this section consists in reducing the dynamic control problems for a certain class of social networks to the standard controllability analysis and control design problems for linear discrete-time control systems. Therefore, further extension of classical control theory methods to social networks seems promising. Also we mention other directions of topical research as follows:

(1) rejecting the very strong assumption (if any) on the strict positivity of the influence matrix; in general, exploring how the communication graph affects the properties and controllability of the social network;
(2) considering control efficiency criteria of general form;
(3) studying control problems for "nonlinear" social networks, i.e., the networks with nonlinear dynamics of the agents' opinions, similar to Eq. (2.5);
(4) introducing two and more Principals that control some (possibly intersecting) sets of agents. If the controls are assumed to be additive, then a standard dynamic game of Principals arises naturally (see the models of informational confrontation in Chap. 3). For this game, we may calculate, e.g., subgame-perfect equilibria [153].
(5) posing and solving control problems for the communications structure of social network members, with further extension and interpretation of the results to the consensus problem [3, 213] and vice versa;
(6) developing simulation models for the dynamic processes of informational control.

2.3 Unified Informational Control in Homogeneous Networks. Role of Mass Media

Assume at an initial time (step) each agent has some *opinion* on a certain issue. The initial opinions of all agents is described by a column vector $x^0 \in \Re^n$. As the result of opinions exchange with the neighbors from a set $N_i = \{j \in N | a_{ij} > 0\}$, agent i changes his/her opinion $x_i^k \in \Re^1$ at step k in accordance with the law

$$x_i^k = \sum_{j \in N_i} a_{ij} x_j^{k-1}, \quad k = 1, 2, \ldots. \tag{2.17}$$

Obviously, any vector composed of identical opinions is a fixed point of map (2.17). Let each agent trust him/herself with some degree: $a_{ii} > 0 \ \forall i$. As the exchange processes are evolving on the infinite time horizon (see Sect. 2.1), the opinion vector of all agents converges to the resulting opinion vector $X = \lim_{k\to\infty} x^k$. If the agents' opinions are stabilized, we may write the relationship

$$X = A^{\infty} x^0, \tag{2.18}$$

where $A^{\infty} = \lim_{k \to \infty} (A)^k$.

This section considers a modification of the Markovian model with homogeneous agents over a connected regular graph. In contrast to model (2.18), it will be supposed below that the opinions of social network members also depend on the messages of *mass media*.

From this viewpoint, the model studied in this section is close to *the imitative behavior model* [206], in which each agent makes binary choice (one of two actions). The difference between the homogeneous social network model and the imitative behavior model is that the former has dynamics and the sets of admissible opinions of all agents are continual.

Homogeneous social network. Trustful agents. Consider the case of a homogeneous network in which all agents have the same initial opinions $x^0 \in \Re^1$ and their communications are described by a connected l-regular graph (i.e., $|N_i| = l, i \in N$).

Besides the agents, the system includes mass media that influence their opinions.

Each agent trusts him/herself with some degree $\alpha \in (0, 1]$, the same for all agents. In addition, each agent trusts mass media with some degree $\beta \in [0, 1]$ ($\alpha + \beta \leq 1$), also the same for all agents. (For example, mass media can be considered as the subject of informational control [165]: from mass media an agent receives information about supposed opinions of the agents who are not directly connected with him/her.) The residual degree of trust $(1 - \alpha - \beta)$ of each agent is equally shared among the agents directly connected with him/her. Mass media report the same opinion $u \in \Re^1$ to all agents. So the opinion dynamics satisfies the recursive equation

$$x_i^k = \alpha x_i^{k-1} + \beta u + \frac{(1 - \alpha - \beta)}{l} \sum_{j \in N_i} x_j^{k-1}, \quad k = 1, 2, \ldots \tag{2.19}$$

Because the social network is homogeneous while the communication graph is regular, relationship (2.19) does not depend on the degree l (i.e., the number of connections of each agent with other agents), on the network size n, and on the degree of trust α [see (2.20)]. Note that Formula (2.20) allows for a probabilistic interpretation: each agent keeps his/her opinion with probability α and accepts the opinion of mass media with probability β.

As the network is homogeneous, we may omit the agent's index, writing

$$x^k = \beta u + (1 - \beta) x^{k-1}, \quad k = 1, 2, \ldots, \tag{2.20}$$

or

$$x^k = u \beta \sum_{\tau=1}^{k} (1 - \beta)^{\tau - 1} + x_0 (1 - \beta)^k, \quad k = 1, 2, \ldots. \tag{2.21}$$

Some elementary transformations of (2.21) yield

$$x^k = u(1-(1-\beta)^k) + x^0(1-\beta)^k, \quad k = 1, 2, \ldots. \tag{2.22}$$

For any step, the agents' opinions belong to the range limited by their initial opinions x^0 and the control u. As $k \to +\infty$ the limit of sequence (2.22) is u.

Interestingly, the exponential curve (2.22) can be treated in terms of learning, memorizing and forgetting of information (see the survey of learning models in [166]).

At each step on a planning horizon, all agents are influenced by an identical control, and this approach is called *unified informational control*. The model under consideration involves constant (time-invariant) unified control, see Formula (2.19). The problem is to find a control law $u(x^*, x^0, T)$ that drives to a desired opinion x^* at a terminal step T under given initial opinions of all agents. Without control constraints, this problem has trivial solution using algebraic transformations of (2.22):

$$u\left(x^*, x^0, T\right) = \frac{x^* - x^0(1-\beta)^T}{1-(1-\beta)^T}. \tag{2.23}$$

As $T \to +\infty$, the control law (2.23) tends to the resulting opinion x^*.

Example 2.19 Choose $\beta = 1/2, x_0 = 0$, and $u = 1$. The dynamics (2.22) of the agents' opinions are demonstrated in Fig. 2.17.

For reaching the opinion $x^* = 1$ at step $T = 10$, the control law (2.23) is $u(1, 0, 10) = 1024/1023$. •

Note that the hypotheses on homogeneous agents and the regular communication graph have actually allowed us **to reduce the whole homogeneous regular social network to a single agent influenced by mass media** [see expression (2.22)]. The essential system parameters—the agent's degree of self-trust, the network size and the regular graph degree—will affect the opinion dynamics only under other relationships differing from (2.19).

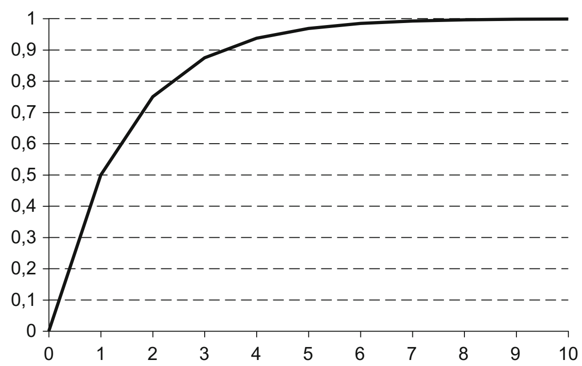


Fig. 2.17 Dynamics of agents' opinions in Example 2.19

Consider a possible law with such properties (in each particular case, a proper choice will be guided by practical interpretations). Let the opinion dynamics be described by

$$x_i^k = \alpha x_i^{k-1} + \beta \frac{(n-l)}{n} u + \frac{(1-\alpha-\beta(n-l)/n)}{l} \sum_{j \in N_i} x_j^{k-1}, \quad k = 1, 2, \ldots. \tag{2.24}$$

Unlike (2.19), in Formula (2.24) mass media reflect the supposed opinion of a part of the social network that is not interacting with a given agent. The share of such agents makes up $\frac{(n-l)}{n}$ and can be treated as *the public opinion weight*.

Again, omitting the agent's index gives

$$x^k = \beta \frac{(n-l)}{n} u + \left(1 - \beta \frac{(n-l)}{n}\right) x^{k-1}, \quad k = 1, 2, \ldots, \tag{2.25}$$

or

$$x^k = u \beta \frac{(n-l)}{n} \sum_{\tau=1}^{k} (1-\beta)^{\tau-1} + x^0 (1 - \beta \frac{(n-l)}{n})^k, \quad k = 1, 2, \ldots. \tag{2.26}$$

After elementary transformations of (2.26), we obtain

$$x^k = u \frac{(n-l)}{n} \left(1 - \left(1 - \beta \frac{(n-l)}{n}\right)^k\right) + x^0 \left(1 - \beta \frac{(n-l)}{n}\right)^k, \quad k = 1, 2, \ldots. \tag{2.27}$$

Note that the opinion dynamics (2.27) in this model are determined by the ratio $\frac{(n-l)}{n}$ rather than by the absolute number connections of each agent with others (the graph degree l). Moreover, the value $u\,\frac{(n-l)}{n}$ is the limit of expression (2.27) as $k \to +\infty$.

The "extreme" cases of Formula (2.27) are the following.

- For $l = n$ (the communication graph is complete), we obtain $x^k = x^0$; so there is no influence of mass media because each agent receives all information from social network members.
- For $l = 0$ (no connections among the agents), mass media have maximal influence and the opinion dynamics are described by (2.22).

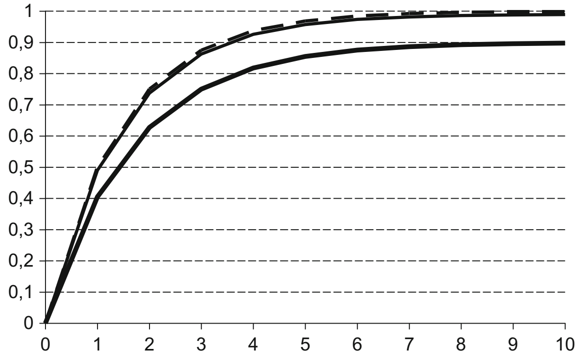


Fig. 2.18 Dynamics of agents' opinions in Examples 2.19–2.20

In this case, an analog of (2.23) is

$$u\left(x^*, x^0, T\right) = \frac{n}{(n-l)} \frac{x^* - x^0\left(1 - \beta\frac{(n-l)}{n}\right)^T}{1 - \left(1 - \beta\frac{(n-l)}{n}\right)^T} \tag{2.28}$$

Example 2.20 Under the data of Example 2.19, consider two communication graphs as follows. In the first graph, $l/n = 0.1$, i.e., each agent is connected with one in ten network members. In the second graph, $l/n = 0.01$, i.e., each agent is connected with one in hundred network members. The opinion dynamics (2.27) are illustrated in Fig. 2.18 (bold line for the first graph). The dashed line in Fig. 2.18 corresponds to the opinion dynamics in Example 2.19.

In accordance with (2.27) (also see Fig. 2.18), under a fixed size of the social network the growing number of agent's connections with other agents actually decreases the influence of mass media (in terms of the variation rate of opinions and also the equilibrium opinion). Conversely, under a fixed degree of network regularity, the growing size of the social network increases the influence of mass media.

For reaching the opinion $x^* = 1$ at step $T = 10$ in the case $l/n = 0.1$, the control law (2.28) is $u(1, 0, 10) \approx 1.114$; in the case $l/n = 0.01$, $u(1, 0, 10) \approx 1.011$. So the higher is the public opinion weight, the smaller is the difference between the mass media message and the agent's opinion for reaching the desired opinion. •

In the model under study, the agents are trusting in the mass media messages with a constant degree regardless of deviations from their own opinions. This scenario corresponds to *trustful agents*. Now, consider another case (*cautious agents*) in which the degree of trust to mass media depends on their messages.

Cautious agents. For making the agent's degree of trust dependent on the mass media messages, introduce a *trust function* $G(x, u)$, where x denotes the agent's opinion and u is control (mass media message). Numerous examples and experimental data can be found in the literature on social psychology, e.g., [152]. The trust function will be assumed to have the following properties (used in different combinations below).

Assumption A.1 The function $G(x, u)$ is nonnegative and achieves the maximum β at $u = x$: $G(x, x) = β$.

Assumption A.2 The function $G(x, u)$ is nonnegative and achieves the minimum β at $u = x$: $G(x, x) = β$.

Assumption A.3 The function $G(x, u)$ depends on the difference $(x - u)$ only.

Assumption A.4 The function $G(x, u)$ is monotonically decreasing in $|x - u|$.

Assumption A.5 The function $G(x, u)$ is monotonically increasing in $|x - u|$.

Assumption A.6 Under Assumptions A.1 and A.3, $\forall x \in \Re^1$ let $\lim_{u \to -\infty} G(x,u) = \beta_-$ and $\lim_{u \to +\infty} G(x,u) = \beta +$, where $β_- \le β$ and $β_+ \le β$. In addition, the function $G(x, u)$ has unique minima on the half-intervals $(-\infty, x]$ and $[x, +\infty)$ of argument u.

Assumption A.7 Under Assumptions A.2 and A.3, $\forall x \in \Re^1$ let $\lim_{u \to -\infty} G(x,u) = \beta_-$ and $\lim_{u \to +\infty} G(x,u) = \beta +$, where $β \le β_-$ and $β \le β_+$. In addition, the function $G(x, u)$ has unique maxima on the half-intervals $(-\infty, x]$ and $[x, +\infty)$ of argument u.

By Assumptions A.1 and A.2, an agent is trusting with maximal (minimal, respectively) degree to the mass media reporting messages that coincides with his/her opinion. By Assumption A.3, the agent's degree of trust to mass media messages depends only on their deviations from his/her opinion, regardless of their values. By Assumptions A.4 and A.5, the closer is a message to the agent's opinion, the higher (lower, respectively) is his/her degree of trust. For example,

$$G(x,u) = \beta \exp(-\gamma|x-u|), \quad \gamma > 0, \tag{2.29}$$

and

$$G(x,u) = 1 - (1-\beta)\exp(-\gamma|x-u|), \quad \gamma > 0. \tag{2.30}$$

By Assumption A.6,

- an agent is trusting with maximal degree to the mass media reporting messages that coincide with his opinion (A.1);
- the greater are the deviations of the mass media messages from the agent's opinion, the lower is his/her trust to them;
- however, under "extreme" messages of mass media, an agent has a higher degree of trust (agents are likely to trust in terrible falsehoods).

For $β_- = β_+ = β$, an example is the function

$$G(x,u) = \beta[1 - (1 - \exp(-\gamma|x-u|))\exp(-\gamma|x-u|)]. \tag{2.31}$$

By Assumption A.7,

- an agent is trusting with minimal degree to the mass media reporting messages that coincide with his opinion (A.2);
- the greater are the deviations of the mass media messages from the agent's opinion, the higher is his/her trust to them;
- however, under "extreme" messages of mass media, an agent has a lower degree of trust (agents are susceptible to conclusions not exceeding their admissible thresholds).

For $\beta_- = \beta_+ = \beta$, an example is the function

$$G(x, u) = (1 - \beta)\exp(-\gamma|x - u|)\exp(-\gamma|x - u|) + \beta. \quad (2.32)$$

The graphs of the trust functions (2.29)–(2.32) are schematically shown in Fig. 2.20.

Thus, there are five cases as follows: (a) $G(x, u) = \beta$; (b) $G(x, u)$ defined by (2.29); (c) $G(x, u)$ defined by (2.30); (d) $G(x, u)$ defined by (2.31); (e) $G(x, u)$ defined by (2.32). In the probabilistic setup (e.g., the trust function is treated as the probability to identify a given message in an informational flow), these cases have the following practical interpretations.

Case 1 (constant trust function). An agent responds to a mass media message regardless of its content.
Case 2 [the trust function of form (2.29)]. An agent is a conservative, i.e., the probability of message identification decreases with the deviation from his/her opinion.
Case 3 [the trust function of form (2.30)]. An agent is an innovator, i.e., the probability of message identification increases with the deviation from his/her opinion.
Case 4 [the trust function of form (2.31)]. An agent is a mild conservative, who identifies the mass media messages coinciding with his/her opinion until the deviation exceeds a sufficiently large threshold. Under large deviations, the probability that he/she identifies such messages is higher.
Case 5 [the trust function of form (2.32)]. An agent is a mild innovator, who identifies mass media messages with higher probability while the deviation from his/her opinion is not very large; yet, for sufficiently large deviations, this probability decreases.

After this brief discussion of practical interpretations of different trust functions, assume that controls can be time-varying. Introduce the following notations: $u^{0,T-1} = (u^0, u^1, \ldots, u^{T-1}) \in \Re^T$ as a sequence of controls; $x^{0,T} = (x^0, x^1, \ldots, x^T) \in \Re^{T+1}$ as a trajectory of social network states; $T \geq 0$ as a planning horizon; $F(x^{0,T}, u^{0,T-1})$ as a control efficiency criterion, where $F(\cdot,\cdot)\colon \Re^{(T+1)T} \to \Re^1$ is a given function. For the time being, there are no control constraints (i.e., they are incorporated in the efficiency criterion).

By analogy with expression (2.20), let the social network states have the controlled dynamics

$$x^k = G\left(x^{k-1}, u^{k-1}\right)u^{k-1} + \left(1 - G\left(x^{k-1}, u^{k-1}\right)\right)x^{k-1}, \quad k = 1, 2, \ldots . \tag{2.33}$$

As a small digression note it seems topical to analyze the following opinion dynamics in an inhomogeneous and irregular social network:

$$\begin{aligned} x_i^k = {} & a_{ii}x_i^{k-1} + \beta\, G_i(x_i^{k-1}, u^{k-1})u^{k-1} \\ & + \sum_{j \in N_i} a_{ij}G_i(x_i^{k-1}, x_j^{k-1})x_j^{k-1}, \quad k = 1, 2, \ldots, \end{aligned} \tag{2.33$'$}$$

where the individual trust functions $\{G_i(\cdot)\}_{i \in N}$ satisfy the normalization condition. Within the framework of this model, matrix A reflects the degrees of trust to *information sources* while the trust functions the degrees of trust in *information content*.

In general form, *optimal informational control design* in a homogeneous social network can be stated as the problem to find a sequence of controls for the dynamic system (2.33) that maximizes the efficiency criterion:

$$F\left(x^{0,T}, u^{0,T-1}\right) \to \max_{u^{0,T-1} \in \Re^T} . \tag{2.34}$$

Problem (2.34) is an optimal control problem and can be solved using well-known methods (see Example 2.21 below). For instance, if the efficiency criterion is additive in the time variable, we may employ Bellman's principle of optimality.

In the case of constant controls, expression (2.33) takes the form

$$x^k = G\left(x^{k-1}, u\right)u + \left(1 - G\left(x^{k-1}, u\right)\right)x^{k-1}, \quad k = 1, 2, \ldots, \tag{2.35}$$

and problem (2.34) can be written as

$$F_0\left(x^{0,T}, u\right) \to \max_{u \in \Re^1} . \tag{2.36}$$

So we have obtained an unconstrained scalar optimization problem in which $F_0(\cdot, \cdot)\colon \Re^{T+1} \to \Re^1$ is a given efficiency criterion with constant controls.

A particular setup of problem (2.34) is as follows. Let x^* be a fixed vector that specifies the goal of informational control. Assume a given function $C(u^{0,T-1})\colon \Re^T \to \Re^1$ defines control cost bounded above by $R \geq 0$. Then problem (2.34) can be written as

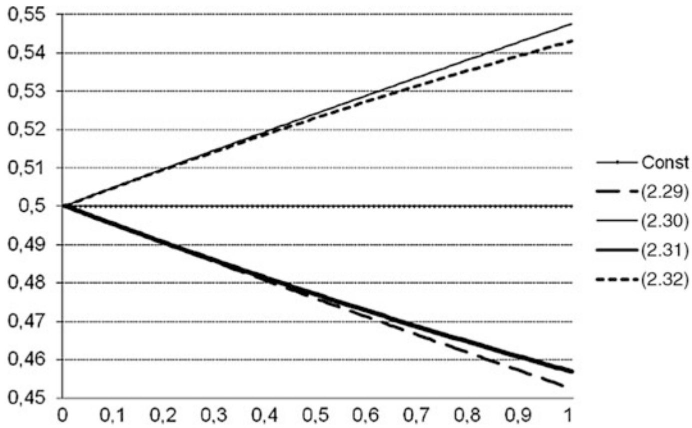


Fig. 2.19 Graphs of trust functions in Example 2.21 ($\gamma = 0.1$)

$$\begin{cases} \|x^T - x^*\| \to \min\limits_{u^{1,T}}, \\ C(u^{0,T-1}) \le R. \end{cases} \tag{2.37}$$

The next example of optimal informational control design well illustrates the relationship between optimal solutions and the properties of trust functions.

Example 2.21 Consider problem (2.37). Choose $\beta = 0.5$, $\gamma = 0.1$, $x_0 = 0$, $x^* = 1$, $T = 10$, $C(u^{0,T-1}) = \sum_{\tau=0}^{T-1} u^\tau$, and $R = 5$, and use the quadratic norm for the goal function of problem (2.37). The trust functions for the five cases above are shown in Fig. 2.19 (the horizontal axis is associated with $|x - u|$).

In accordance with Fig. 2.19, under small values of the parameter γ the trust functions in this example have almost linear character; moreover, the graphs of the trust functions (2.29) and (2.31) [(2.30) and (2.32)] almost coincide with each other. For higher γ, they are differing more and more, see Fig. 2.20 with the graphs for $\gamma = 3$.

The constant controls in all the cases are 0.5.

Figures 2.21 and 2.22 demonstrate the opinion dynamics with the trust functions (2.29), (2.30), (2.31) and (2.32) under the optimal constant controls for $\gamma = 0.1$ and $\gamma = 3$, respectively. Due to the control constraints, the agents' opinions cannot reach the goal value $x^* = 1$ as desired.

Figures 2.23 and 2.24 present the graphs of the degrees of trust with the trust functions (2.29), (2.30), (2.31) and (2.32) under the optimal constant controls for $\gamma = 0.1$ and $\gamma = 3$, respectively.

Now, consider a more complex scenario with variable controls—a special case of problem (2.34) defined by (2.33), (2.37). This is a linear discrete problem with a quadratic integral criterion over a fixed time horizon.

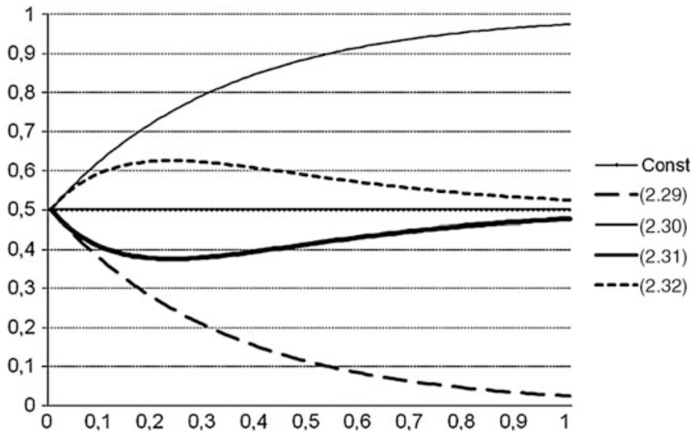


Fig. 2.20 Graphs of trust functions in Example 2.21 ($\gamma = 3$)

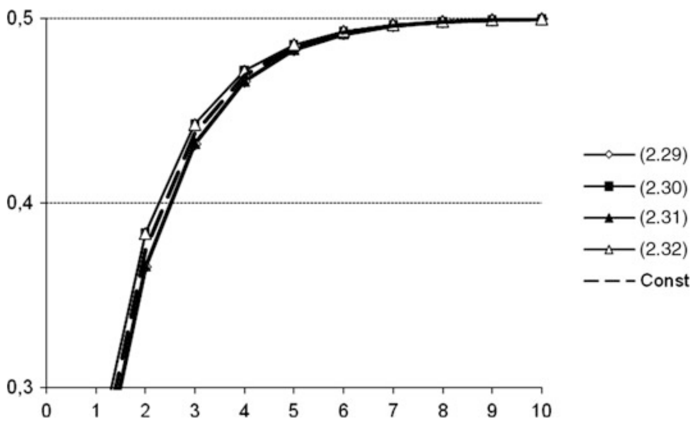


Fig. 2.21 Opinion dynamics under optimal constant controls in Example 2.21 ($\gamma = 0.1$)

Figures 2.25 and 2.26 show the opinion dynamics with the trust functions (2.29), (2.30), (2.31) and (2.32) under the optimal variable controls for $\gamma = 0.1$ and $\gamma = 3$, respectively.

Figures 2.27 and 2.28 present the graphs of the degrees of trust with the trust functions (2.29), (2.30), (2.31) and (2.32) under the optimal variable controls for $\gamma = 0.1$ and $\gamma = 3$, respectively.

The graphs of the optimal variable controls for $\gamma = 0.1$ and $\gamma = 3$ are given in Fig. 2.29 and Fig. 2.30, respectively.

The values of the efficiency criterion (to be minimized) are combined in Table 2.1. •

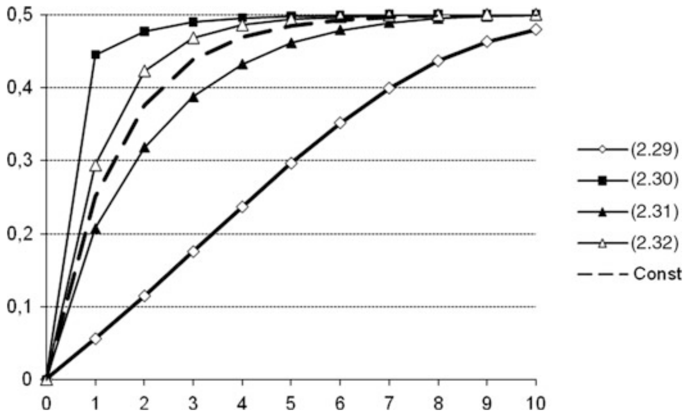


Fig. 2.22 Opinion dynamics under optimal constant controls in Example 2.21 ($\gamma = 3$)

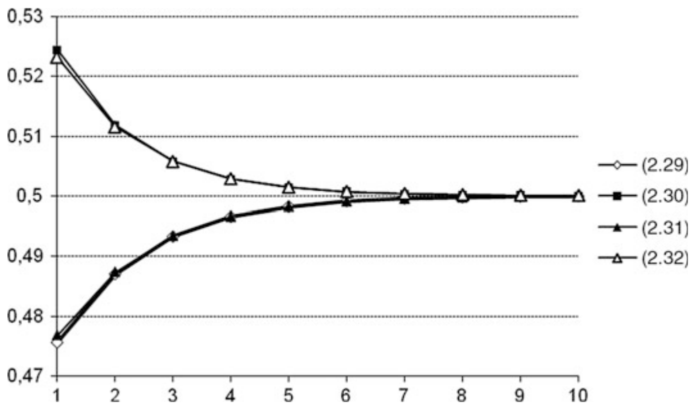


Fig. 2.23 Dynamics of degrees of trust under optimal constant controls in Example 2.21 ($\gamma = 0.1$)

The main result of this section at qualitative level is as follows. We have reduced unified informational control design in homogeneous social networks described by regular communication graphs to the dynamic analysis of single agent's opinions under the informational influence of mass media. It would be interesting to consider how the agent's degree of trust in mass media messages depends on the content of reported information rather than on its source (which is traditionally described by the Markovian models of social networks, see Sect. 2.4). In other words, a promising line of further research is to examine how the deviations between the messages and agent's beliefs affect his/her degree of trust.

It seems that the rather strong assumptions above (communication graph regularity and agents; homogeneity) have allowed us to obtain simple analytic

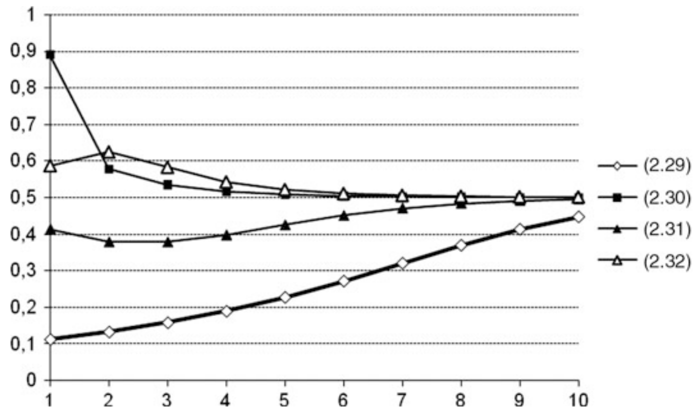


Fig. 2.24 Dynamics of degrees of trust under optimal constant controls in Example 2.21 ($\gamma = 3$)

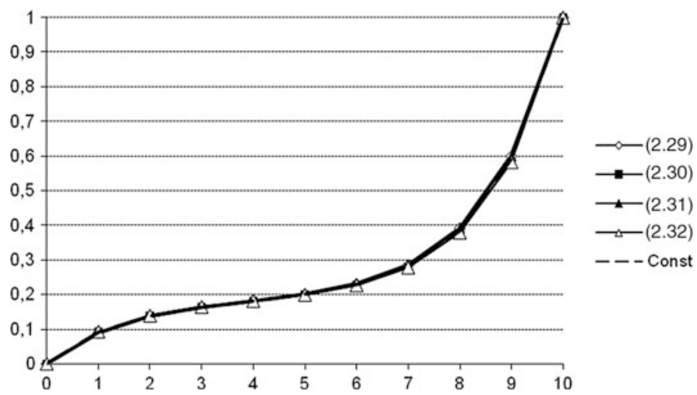


Fig. 2.25 Opinion dynamics under optimal variable controls in Example 2.21 ($\gamma = 0.1$)

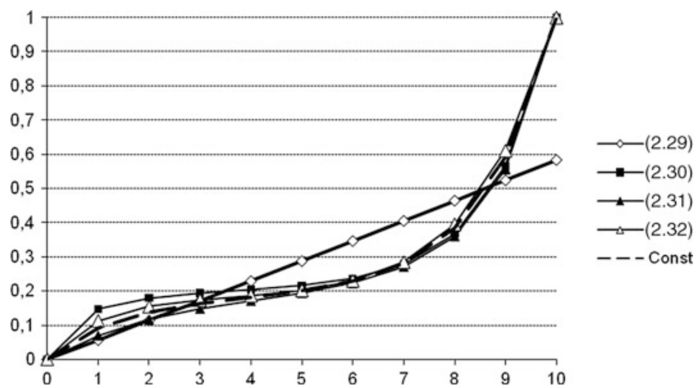


Fig. 2.26 Opinion dynamics under optimal variable controls in Example 2.21 ($\gamma = 3$)

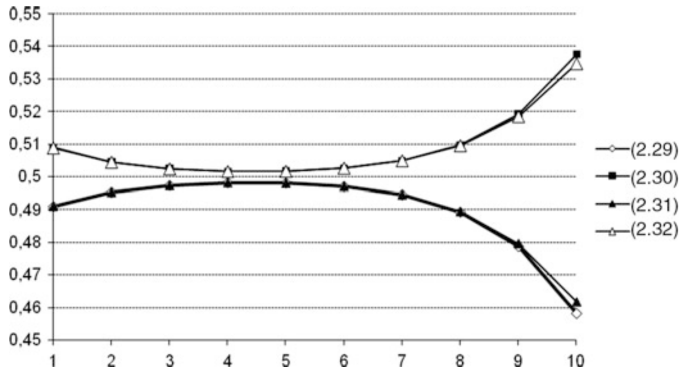


Fig. 2.27 Dynamics of degrees of trust under optimal variable controls in Example 2.21 (γ = 0,1)

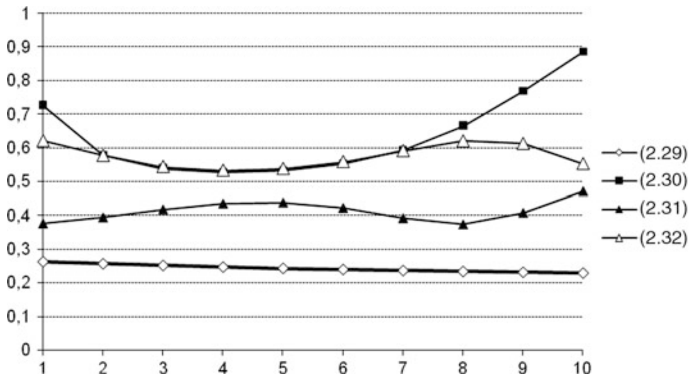


Fig. 2.28 Dynamics of degrees of trust under optimal variable controls in Example 2.21 (γ = 3)

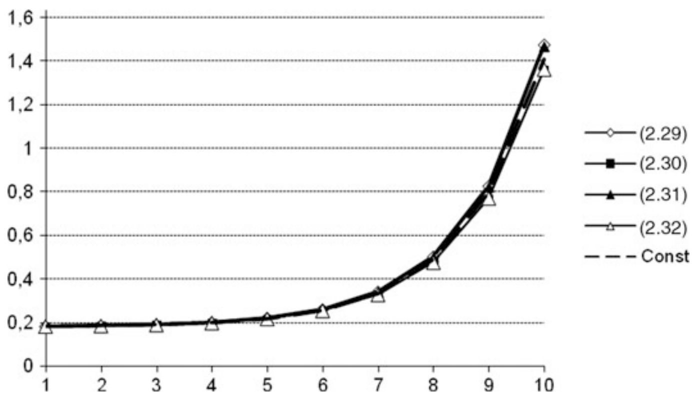


Fig. 2.29 Optimal variable controls in Example 2.21 (γ = 0.1)

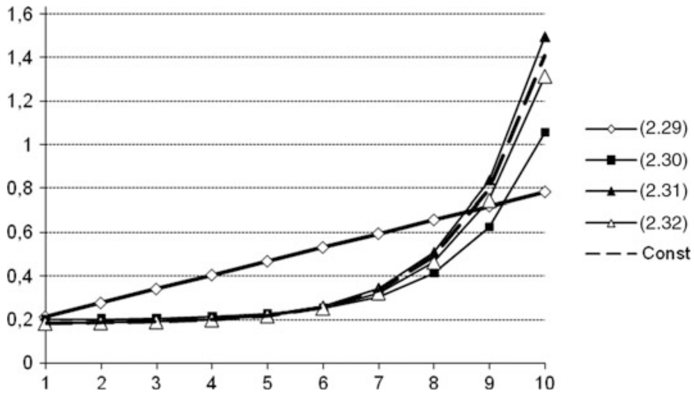


Fig. 2.30 Optimal variable controls in Example 2.21 ($\gamma = 3$)

Table 2.1 Values of efficiency criterion

Case	Efficiency of constant controls		Efficiency of variable controls	
	$\gamma = 0.1$	$\gamma = 3$	$\gamma = 0.1$	$\gamma = 3$
1: $G(\cdot) = \beta$	0.2505	0.2505	0	0
2: $G(\cdot)$ defined by (2.29)	0.2505	0.2711	0	0.1736
3: $G(\cdot)$ defined by (2.30)	0.2504	0.25	0	0
4: $G(\cdot)$ defined by (2.31)	0.2505	0.2515	0	0
5: $G(\cdot)$ defined by (2.32)	0.2504	0.2502	0	0

expressions for opinion dynamics and to reduce informational control design to well-known optimization problems.

Another topical field of investigations is to describe and study the nonlinear models of social networks with complex trust in which the agent's degree of trust to his/her neighbor depends not only on the source but also content of information reported—see (2.33‘). Though, in the general case of inhomogeneous agents, one can hardly expect simple analytic formulas [like (2.18)] for the equilibrium states of social networks.

In the class of threshold models of social networks, there exists a series of publications with rigorous statements and explicit solutions of informational control design, in discrete or continuous time [12, 13, 29, 167].

It is possible to consider other (e.g., threshold) classes of trust functions, to complicate the agent's internal structure (by analogy with the bipolar choice models [168] or Lefevbre's logical models [134]). The model can be also generalized using reflexion: agents choose actions depending on their opinions and observe the results of these actions (the "opinion–action–result" approach). Then, in addition to efficiency, the stability problems of informational influences [168] arise naturally.

Finally, non-Markovian dynamics of agents' opinions can be studied, e.g., each agent tries to predict the opinion variations of other agents, etc. But such setups proceed from the hypothesis that the whole social network is common knowledge among all agents, actually a very strong assumption. All these are the subjects of future research on informational control modeling for social networks.

2.4 Informational Control and Reputation of Network Members

Reputation. In a social network, the reputation of a given member to a large extent predetermines his/her capabilities of influencing other members. In accordance with the Merriam–Webster Dictionary, reputation is an overall quality or character as seen or judged by people in general. Reputation may be regarded, first, as an expected norm of the agent's activity—which behavior is expected from him/her by other agents [60]. Second, as a "weight" of the agent's opinion determined by past verifications of his/her opinions and/or efficiency of his/her activity. Reputation is justified and often grows if the agent's choice (opinions, actions, etc.) coincides with what the others expect from him/her and/or with what the others consider as the norm (e.g., efficient activity). Reputation may also decline, e.g., when an agent violates behavioral standards of the community, when he/she makes an inefficient decision, etc. Note that there exist individual and collective reputation. The models of individual and collective reputation were surveyed in [60].

Let $r_i \geq 0$ be a parameter that describes the *reputation* of agent i. Unless otherwise stated, the vector of all reputations $r = (r_1, r_2, \ldots, r_n)$ forms common knowledge among the agents. Assume an agent with nonzero reputation always exists in the network, which is a complete graph. In accordance with the outcomes of Sect. 2.1, the resulting opinion will be the same for all agents in the social network.

Define the degree of trust of agent i in agent j as

$$\alpha_{ij} = \frac{r_j}{\sum_{k \in N} r_k}, \quad i,j \in N. \tag{2.38}$$

In other words, the influence level of each agent does not depend explicitly on the objects of influence, being proportional to his/her relative reputation. As follows from (2.38), agent i has greater susceptibility to the influence of agent j if the latter's reputation is higher while the reputations of the former and all other network members are lower.[5]

[5]Naturally, the relationship between the influence level and reputation can be defined in another way, with the properties of partial monotonicity and transparent interpretations in applications (see the discussion below).

Note that the normalization condition (2.1) always holds for the degree of trust (2.38). Denote by $R = \sum_{k \in N} r_k$ *the total* ("*collective*") *reputation* of all network members.

Then the agents' opinions have the linear dynamics

$$x_i^\tau = \frac{1}{R} \sum_{j \in N} r_j x_j^{\tau-1}, i \in N, \tag{2.39}$$

while the resulting opinion of all agents is given by

$$X = \frac{1}{R} \left(r \cdot x^0 \right). \tag{2.40}$$

Therefore, the scalar resulting opinion X (the same for all agents!) is defined by the scalar product of the reputation vector r and the initial opinion vector x^0, with normalization by the total reputation. Interestingly, the resulting opinion is formed in one step.

Manipulation of opinions for social network members. The elementary model of *informational control* (manipulation of opinions in a social network[6]) is the following. Assume a certain agent (without loss of generality, agent 1 with a reputation $r_1 > 0$) is interested in a resulting opinion X_*. For a given reputation vector and fixed opinions of the other agents, this can be achieved [see (2.40)] by reporting

$$s_1 = \frac{1}{r_1} \left[R X_* - \sum_{k > 1} r_k x_k^0 \right]. \tag{2.41}$$

Using the nonnegativity of all initial opinions (particularly, $x_1^0 \geq 0$), we may find a lower limit for the "manipulation range" of agent 1:

$$X_* \geq \frac{1}{R} \sum_{k > 1} r_k x_k^0. \tag{2.42}$$

(With unbounded messages and nonzero reputation, he/she may further improve this value as much as needed.)

In accordance with Formula (2.42), **the higher is the reputation of the agent performing manipulation, the wider are his/her capabilities to influence the resulting opinion of the agents in a social network**.

[6]Under manipulation we understand a purposeful formation of opinions for social network participants, i.e., informational control. A negative connotation of this term is not supposed, i.e., manipulation is considered ethically neutral. The second (close) meaning of the term "manipulation" is the distortion (mispresentation) of any information reported by an agent (also see below).

In the general case, all the agents can manipulate the resulting opinion by reporting different opinions than their true ones. This leads to the model of linear *active expertise*[7] [see expression (2.40)], which is well-known in the literature [165].

Now, let us study the manipulation capabilities of agent 1 depending on his/her reputation. Assume agent 1 may report an initial opinion only above some bound $x_1^{\min} > 0$. Then the minimal reputation of agent 1 for achieving the equilibrium X_* under the constraint $x_1^{\min} > 0$ can be calculated by

$$r_1 = \frac{\sum_{j>1} r_j (x_j^0 - X_*)}{X_* - x_1^{\min}} \tag{2.43}$$

From Formula (2.43) it follows that the higher is the reputation of other agents, the stronger are the requirements applied to the reputation of manipulating agent.

In real social networks, agents can often report their opinions within a rather wide range. However, as a rule they cannot choose their reputation by themselves because it significantly depends on the history of agents' interaction.

Further considerations involve the following idea at qualitative level. If a certain agent wants to manipulate the opinions of social network members, then he/she needs sufficient reputation. Therefore, it is necessary to examine a scenario in which an agent first undertakes some actions to increase his/her reputation and then uses it for individual goals—efficient manipulation. So the problem consists in a proper description of (1) reputation dynamics and (2) reputation development processes for certain purposes.

Reputation dynamics. For the dynamic modeling of the agents' reputation, assume their interaction—see the previous paragraph—is repeated sequentially a finite number of times, with different initial conditions. In a practical interpretation, the agents may sequentially discuss several issues of interest, and the reputation of each agent generally depends on the whole history of discussions.

There are T sequential time periods, and the members of a social network are sequentially considering each of T issues at a corresponding step. Each agent has an initial opinion x_i^τ, $i \in N, \tau = \overline{1, T}$, on each of these issues. Denote by r_i^1, $i \in N$, the initial reputations of the agents. Assume the common knowledge of the agents covers their reputations (initial and current reputations as well as the whole history of reputation dynamics), the initial and resulting opinions of all agents for the current and all past periods.[8]

[7]The process of exchanging opinions among social network members that results in some collective opinion can be interpreted as expertise.

[8]The operation of a social network can be considered in two time scales, fast (in which the opinions of all network members on a fixed issue are converging) and slow (in which the network members are sequentially discussing different issues).

Denote by R^τ the total reputation of the agents at the beginning of period τ and by X^τ the resulting opinion of the agents at the end of this period. As follows from (2.40), this opinion will be the same for all the agents.

Thus, the issues considered by the agents are independent and the resulting opinions will be given by

$$X^\tau = \frac{1}{R^t}(r^\tau \cdot x^\tau), \tag{2.44}$$

where $r^\tau = (r_1^\tau, \ldots, r_n^\tau)$ and $x^\tau = (x_1^\tau, \ldots, x_n^\tau)$ are the vectors of reputations and initial opinions of the agents at the end of period τ, $\tau = \overline{1, T}$.

In order to describe the whole trajectory of opinions and reputations of the agents, it is necessary to determine how the reputation of each agent changes in each period. Assume reputation is a cumulative characteristic (no forgetting), and the reputation of any agent at the beginning of each period coincides with his/her reputation at the end of the previous period.

The issues considered by the agents belong approximately to the same theme; so an agent with high reputation on one issue (as the result of discussions) will have the same reputation as the debates proceed to the next issue.

In the general case, we may hypothesize that the reputation of agent i in period τ is defined by the initial and resulting opinions of all the agents and their reputations in all the previous periods:

$$r_i^\tau = F_i(r^1, \ldots, r^{\tau-1}, x^1, \ldots, x^{\tau-1}, X^1, \ldots, X^{\tau-1}), \quad i \in N,\ \tau = \overline{2, T}. \tag{2.45}$$

(By assumption, each agent prefers truth-telling and reports reliable information.) Moreover, it seems reasonable to expect that the function $F_i(\cdot)$ is at least monotonically decreasing in the difference $\left|x_i^{\tau-1} - X^{\tau-1}\right|$ and also increasing in the previous reputations of agent i. For example, the following law of reputation variation can be used:

$$r_i^\tau = \frac{r_i^{\tau-1}}{\gamma + \beta |x_i^{\tau-1} - X^{\tau-1}|}, \quad i \in N,\ \tau = \overline{2, T}, \tag{2.46}$$

where $\gamma \in (0,\, 1]$ and $\beta > 0$ are given constants. In accordance with Formula (2.46), the agent's reputation at the beginning of any period depends on his/her reputation in the previous period and also on how much his/her initial opinion in the previous period turned out to be different from the resulting opinion of all agents at the end of this period. In other words, the agent's reputation is increasing (decreasing), and the rate of variation is determined by the constants γ and β if the resulting opinion of all the agents turns out to be close to (considerably differs from) his/her opinion.

The law of reputation variation (2.46) is a possible one. For example, the logistic law of reputation variation [60] or other approaches are often used: in each particular case, it is necessary to perform identification—find the best laws for the observed or predicted effects.

Hopefully, complex dynamic models of reputation will provide a good description for many frequently encountered phenomena such as untrue reputation, inertia of reputation dynamics (by ceasing "investments" into his/her own reputation, an agent may still take advantage of it for some time), and others (see examples in [60]). The development of such game-theoretic models is a long-term goal of further research that goes beyond the scope of this book.

Following the description of informational influence and reputation dynamics, let us state and solve an associated control problem.

Informational control problem. With Eqs. (2.44) and (2.45) that model the opinion dynamics depending on reputation and the reputation dynamics depending on the opinion dynamics, we can formulate and solve an associated *control* problem —calculate an influence on social network agents that leads to the formation of required opinions.

Further analysis will be confined to the case of manipulation performed by a single agent (agent 1), who tries to manipulate his/her initial opinions on each issue for achieving a certain resulting opinion of all network members on the last question (through appropriate dynamics of his/her reputation).

Thus, we have the dynamic system (2.44)–(2.45). It is required to find a sequence $s_1^1, s_1^2, \ldots, s_1^T$ of the initial opinions of agent 1 reported to the other agents that satisfies the constraints $s_1^\tau \geq x_1^{\tau\min}$, $\tau = \overline{1, T}$, and also minimizes a given monotonic goal function $F\left(\left|X^T - X_*^T\right|\right)$. (Manipulation actually consists in reporting $s_1^\tau \neq x_1^\tau$, and a desired resulting opinion X_*^T on the last issue can be interpreted as the goal of control—manipulation).

In the general case, this problem is of dynamic programming (under appropriate constraints imposed on the properties of the functions and admissible sets) and may be solved numerically in each particular case.

Consider the following behavioral *heuristics* for agent 1. Recall that the higher is the reputation of the agent performing manipulation, the wider are his/her opportunities of influencing the resulting opinions of all agents in a social network with fixed reputations. Thus, for agent 1 it is desirable to have the maximum possible reputation by the beginning of the last period. Let the function $F_1(\cdot)$ satisfy the monotonicity condition above and be such that the reputation of agent 1 in the current period depends only on his/her reputation in the previous period, on his/her initial opinion in the previous period and on the resulting opinion in the previous period [further referred to as Assumption (*)]. In this case, consider the following solution of the informational control problem: in each period except the last, agent 1 should independently choose an initial opinion to maximize his/her reputation at the end of this period. In the last period, agent 1 should choose his/her initial opinion by minimizing $F\left(\left|X^T - X_*^T\right|\right)$ under the established and fixed reputation, and the value X^T will depend only on the initial opinion s_1^T in period T.

Formally, agent 1 solves a system of problems that includes $T - 1$ independent reputation maximization problems and a single choice problem of his initial opinion in the last period:

$$\left| s_1^\tau - \frac{1}{R^\tau}\left[r_1^\tau s_1^\tau + \sum_{j>1} r_j^\tau x_j^\tau \right] \right| \to \min_{s_1^\tau \geq x_1^{\tau\min}}, \quad \tau = \overline{1, T-1}, \tag{2.47}$$

$$\left| \frac{1}{R^\tau}\left[r_1^T s_1^T + \sum_{j>1} r_j^T x_j^T \right] - X_*^T \right| \to \min_{s_1^T \geq x_1^{T\min}}. \tag{2.48}$$

Without any constraints on the initial opinions reported by agent 1, the solution of problem (2.47) has the form

$$s_1^\tau = \frac{\sum_{j>1} r_j^\tau x_j^\tau}{\sum_{j>1} r_j^\tau}, \quad \tau = \overline{1, T-1}. \tag{2.49}$$

So agent 1 is maximizing his/her reputation by expressing "weighted average" opinion of the other agents considering their reputations. Figuratively speaking, Formula (2.49) illustrates the principle "always say what the majority does and be counted wise."[9]

Thus, during the first $T-1$ periods, the manipulating agent is maximizing his/her reputation, and in the last period uses the latter for achieving the goals of informational control. Although looking quite rational, such behavior is merely a heuristic yielding no accurate solution of the informational control problem. The cause is that the total reputation of all agents appears in the resulting reputation R^T in period T [see Formula (2.48)] but, in each period, agent 1 ignores this fact and chooses his/her actions by rule (2.48), thereby influencing the reputation of the other agents [see Assumption (*)]. This phenomenon is also illustrated by an example in Sect. 3.1. The heuristic solution can be transformed into the exact one by defining the influence and reputation so that the resulting reputation is constant,[10] or using *the hypothesis of weak contagion* [165].

Fuzzy model of social network. The social network model under consideration (see the current paragraph and also Sect. 2.1), which reflects the informational influence of agents, their reputation and opinion dynamics, can be called *the basic model of a social network*. Now, extend it to the fuzzy case.

A rather simple form of expression (2.40), which describes the relationship between the resulting opinion of social network members and their initial opinions and reputations, allows us to obtain a similar formula in the case where the reputations and initial opinions of all agents are fuzzy. Such a generalization will be called *the fuzzy model of a social network*.

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[9]To be more exact, this formula implies prediction for opinions exchange.

[10]Normalization of individual reputations by the total one leads to a Markovian model in which the probabilities of steady states are defined by the relative reputations of corresponding agents. A steade state is a collective decision that coincides with the opinion of some agent.

Assume the fuzzy initial opinion of agent i is described by a membership function $\nu_i(x_i)$: $[0; +\infty) \rightarrow [0; 1]$, $i \in N$. Let the reputations of all agents be fuzzy with some membership functions $\mu_i(r_i)$: $[0; +\infty) \rightarrow [0; 1]$, $i \in N$.

Following the principle of generalization [173], we may write the following expression for the membership function of the fuzzy initial opinions of all agents in a social network:

$$\mu(X) = \max_{\left\{(r,x)\mid \frac{\sum_{j\in N} r_j x_j}{\sum_{i\in N} r_i} = X\right\}} \min_{i\in N}\{\min[\mu_i(r_i); \nu_i(x_i)]\}.$$

This transition from the basic model to its fuzzy analog naturally satisfies the principle of conformity: in the "limiting" case (the reputations and initial opinions of all agents are crisp), the above formula of $\mu(X)$ gives the same result as (2.39).

Example 2.22 Two agents have crisp reputations and fuzzy initial opinions defined on a binary support —the set $\{0; 1\}$—with the membership functions $\nu_1(0) = 1 - p$, $\nu_1(1) = p$, $\nu_2(0) = 1 - q$, and $\nu_2(1) = q$, where $p, q \in [0; 1]$.

Then

$$\mu(X) = \max_{\left\{(x_1,x_2)\mid \frac{r_1 x_1 + r_2 x_2}{r_1 + r_2} = X\right\}} \min[\nu_1(x_1); \nu_2(x_2)]\}.$$

Thus, the resulting opinion is a fuzzy variable $\tilde{X}$ with the finite support $\left\{0; \frac{r_2}{r_1 + r_2}; \frac{r_1}{r_1 + r_2}; 1\right\}$ and the membership function that takes the corresponding values (min $[(1 - p); (1 - q)]$; min $[(1 - p); q]$; min $[p; (1 - q)]$; min $[p; q]$).

If $p = 1/3$, $q = 1/4$, $r_1 = 1$, and $r_2 = 2$, then the fuzzy resulting opinion of social network members is $\{0|2/3;\ 1/3|1/3;\ 2/3|1/4;\ 1|1/4\}$. In Fig. 2.31, the values of the membership function are set in bold type. •

This example well illustrates a remarkable property of the fuzzy model: even for the same carriers of fuzzy initial opinions of agents, the carrier of their resulting opinion may differ. This property plays crucial role for informational control problems and their solvability even if the initial opinions of all agents are finite and pairwise distinguishable. •

Informational confrontation. Now, suppose a part of the agents—further called *active*—can perform manipulation by choosing at each step the opinions (messages) reported to other agents from a given value set. (In a more complicated setup, they make choice at a specified step.) Of course, active agents are considering the impact of these messages on the resulting opinions and also on reputation dynamics. The preferences of active agents are defined over the set of all sequences of resulting opinions of social network members on issues of interest. It is required to solve the game of active agents, i.e., to find the sets of their equilibria in some sense. A proper concept of equilibrium is predetermined by practical interpretations as well as by the sequence and amount of information received by all agents. For

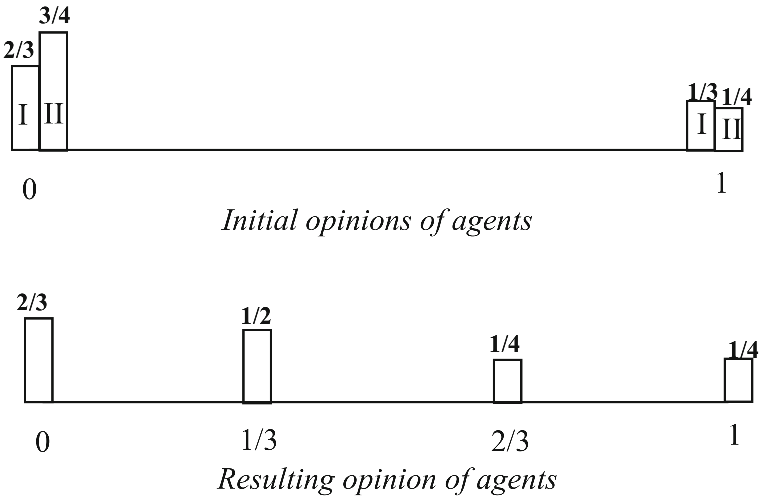

Fig. 2.31 Initial and resulting opinions of agents in Example 2.22

example, we may consider repeated, extensive form, cooperative and other games over social networks.

Within the framework of the suggested social network model, the problem of informational confrontation is actually reduced to *the problem of dynamic active expertise with reputation*, which seems to be a fruitful generalization of classical collective choice problems. Dynamic active expertise with reputation inevitably leads to the problem of *strategy proofness*, a central one for collective choice theory, which can be stated as the following question. Which decision procedures (mutual informational influence processes of agents) guarantee that truth-telling (reporting actual opinions) is preferable to manipulation? This question still has no answer.

Example 2.23 Consider an interaction of three agents ($n = 3$) during two periods ($T = 2$). The initial opinions of the agents are $x_1^1 = 1$, $x_2^1 = 2$, $x_3^1 = 3$, $x_1^2 = 4$, $x_2^2 = 5$, $x_3^2 = 6$, $x_i^{\tau\min} = 0.5$, $i = 1, 2, 3, \tau = 1, 2$. They have the same initial reputations, $r_1^1 = r_2^1 = r_3^1 = 1$, and the reputation dynamics are given by (2.46) with $\gamma = 1/2$ and $\beta = 1$.

First, calculate the resulting opinions and reputations without manipulation when all agents report actual information. The total reputation in period 1 is $R^1 = 3$. Using (2.40) find $X^1 = 2$. By Formula (2.46) the reputations of the agents in period 2 are $r_1^2 = 2/3$, $r_2^2 = 2$, and $r_3^2 = 2/3$. Again using (2.40), calculate the resulting opinion of the agents at the end of period 2: $X^2 = 5$.

Now, let agent 1 perform manipulation for achieving the same resulting opinion in period 2 as his/her own opinion, i.e., $X_*^2 = x_1^2$ (this goal function has similar interpretation as in the models of active expertise [165]). To this effect, he/she should choose two values, $s_1^1, s_1^2 \geq x_1^{\min} = 0.5$, that minimize the following goal function [see (2.48)]:

$$F\left(\left|X^T - X_*^T\right|\right) = \left|\frac{1}{R^2}\left[r_1^2 s_1^2 + r_2^2 x_2^2 + r_3^2 x_3^2\right] - X_*^T\right|. \tag{2.50}$$

From Formula (2.40) it appears that $X^1(s_1^1) = (s_1^1 + 5)/3$. Substituting expression (2.46), we find the relationships between the reputations of all agents in period 2 and the actions of agent 1 in period 1:

$$r_1^2(s_1^1) = \frac{6}{3 + 2\left|2s_1^1 - 5\right|}, \quad r_2^2(s_1^1) = \frac{6}{3 + 2\left|1 - s_1^1\right|},$$
$$r_3^2(s_1^1) = \frac{6}{3 + 2\left|4 - s_1^1\right|}.$$

In final analysis, problem (2.50) takes the form

$$\left|\frac{r_1^2(s_1^1)s_1^2 + 5r_2^2(s_1^1) + 6r_3^2(s_1^1)}{r_1^2(s_1^1) + r_2^2(s_1^1) + r_3^2(s_1^1)} - 4\right| \to \min_{s_1^1 \geq 1/2, s_1^2 \geq 1/2}. \tag{2.51}$$

The solution is $s_1^1 = 2.5$, $s_1^2 = 2.5$ (in period 2, the reputations are $r_1^2 = 2$, $r_2^2 = 1$, $r_3^2 = 1$). The goal function (2.51) takes value 0, which means that the goal of control is completely reachable under the existing constraints $\left(X^2 = 4 = 4 = X_*^2\right)$. Interestingly, in this example the heuristic algorithm gives the optimal solution.

Next, consider the case in which agents 1 and 2 both perform manipulation, the former for achieving the same resulting opinion in period 1 as his/her own opinion, $X_*^1 = x_1^1$, whereas the latter for achieving the same resulting opinion in period 2 as his/her own opinion, $X_*^2 = x_2^2$. Then $X^1\left(s_1^1, s_2^1\right) = \left(s_1^1 + s_2^1 + 3\right)/3$. Find the relationships between the reputations of all agents in period 2 and the actions of agents 1 and 2 in period 1:

$$r_1^2\left(s_1^1, s_2^1\right) = \frac{6}{3 + 2\left|2s_1^1 - s_2^1 - 3\right|}, \quad r_2^2\left(s_1^1, s_2^1\right) = \frac{6}{3 + 2\left|2s_2^1 - s_1^1 - 3\right|},$$
$$r_3^2\left(s_1^1, s_2^1\right) = \frac{6}{3 + 2\left|6 - s_1^1 - s_2^1\right|}.$$

Agent 1 should choose s_1^1 and minimize his/her goal function $F\left(X^1 - x_1^1\right) = \left|\frac{1}{3}\left[s_1^1 + s_2^1 + 3\right] - 1\right| = \left|\frac{1}{3}\left[s_1^1 + s_2^1\right]\right|$ subject to the existing opinion constraints. As easily verified, regardless of the opponent's actions the minimum value is achieved at $s_1^1 = 0.5$.

Agent 2 chooses s_2^1 in period 1 to maximize his/her reputation. To this end, he/she should minimize $\left|s_2^1 - \frac{1}{3}\left[s_1^1 + s_2^1 + 3\right]\right|$ subject to the opinion constraints. As a result, $s_2^1 = 1.75 = X^1$ (so agent 1 has not completely achieved his/her goal, 1.75 − 1.0 = 0.75). The reputations of all agents in period 2 are $r_1^2 = 4/7$, $r_2^2 = 2$, and $r_3^2 = 4/7$.

In period 2, agent 2 should choose s_2^2 and minimize his/her goal function

$$F(X^2 - x_2^2) = \left|\frac{4r_1^2(s_1^1, s_2^1) + s_2^2 r_2^2(s_1^1, s_2^1) + 6r_3^2(s_1^1, s_2^1)}{r_1^2(s_1^1, s_2^1) + r_2^2(s_1^1, s_2^1) + r_3^2(s_1^1, s_2^1)} - 5\right|$$

subject to the opinion constraints. Consequently, $s_2^2 = 5$, which indicates that the goal of agent 2 is completely achieved. •

In fact, other games with a fixed sequence of moves can be considered by analogy.

Reflexion of agents. In accordance with the hypothesis above, such social network parameters as the initial opinions of each agent on each issue, the reputations of all agents, the formation law of the resulting opinions and also reputation dynamics are common knowledge of all agents. However, in real applications this hypothesis may fail: for example, in large social networks the agents do not know all members while the beliefs of agents about the opinions and/or reputations of each other can be incomplete and/or differ. Such situations are well described using uncertain factors (incomplete awareness) and/or nontrivial mutual awareness of agents. For informational control problems in social networks, uncertainty can be introduced by analogy with other decision models and game-theoretic models [168]. So we will discuss in brief the reflexion of agents.

Along with *informational reflexion* based on the asymmetric awareness of agents, it seems interesting to study *strategic reflexion*, traditional for game-theoretic models. This is the process and result of agent's thinking of the possible actions to be chosen by opponents. An important remark should be made here as follows. Within the model under consideration, the agents are not active participants because they choose no actions and have no personal preferences. They merely form their own opinions using the opinions of others in the passive mode (based on trust). The only exception is an agent who performs manipulation: he/she plays with a definite goal and chooses an optimal action for achieving it. In other words, common agents and manipulator are two fundamentally different objects of modeling. Their distinction is not transparent in simple cases (see the previous example) but crucial in complex scenarios (e.g., informational confrontation of several manipulators). Once again, in contrast to a common agent who changes his/her opinion depending on the opinions of others, a manipulator forms the opinions of others (not changing his/her own opinion) for definite goal. So the network nodes are considered as *agents* controlled by *players* of "high intellect" (particularly, a player can be an agent or a group of agents).

Perhaps, there are two possible approaches to model players. The first approach employs the idea that the players are not social network elements (agents) and merely influence the network in some way (such an approach is adopted in Sects. 2.3 and 3.1). The second approach treats the players as agents (social network elements) for which the reputations of other agents are totally insignificant: they never change their opinions. A detailed exploration of both approaches goes beyond the scope of this book.

For the sake of illustration, consider a decision model in which an arbitrary agent $i \in N$ chooses messages to be reported as his/her opinion to other agents[11] (the case of strategic reflexion [168]). Let this agent be interested in that the resulting opinion coincides with his/her message.[12] In a practical interpretation, this agent has authority (high reputation) among opponents—the whole community "agrees" with him/her.

In the case of no reflexion, by (2.40) agent i will report [also see (2.49)]

$$s_i^*(r, x_{-i}) = \frac{\sum_{j \neq i} r_j x_j^0}{R - r_i}, \tag{2.52}$$

where $x_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $i \in N$. Expression (2.52) means that, with this decision rule, the agent ignores his/her opinion and reports the weighted average opinion of all other agents (the weight coefficients are their reputations). Vector (2.52) can be called *the reflexive equilibrium of rank 1*, see the details below.

Which assumptions on the opponents' decision-making are made by the agent? If each agent used a decision rule like (2.52), the only "equilibrium" would be the same opinion reported by all agents. Moreover, if the agents had the same reputations, this would be a Nash equilibrium only under the same actual opinions of all agents.

Therefore, introduce the factor of strategic reflexion in the following way. Assume agent i chooses his/her message (2.52), expecting that all other agents **prefer truth-telling**. (In view of the discussed distinctions of agents and manipulators, this assumption means that the agents are players of "moderate intellect".[13]) If all agents behave in this way, the resulting opinion is

$$\hat{X} = \frac{1}{R} \sum_{i \in N} \frac{\sum_{j \neq i} r_j x_j^0}{R - r_i} r_i. \tag{2.53}$$

In the case of two agents, expression (2.53) takes the form $\hat{X} = \frac{x_1^0 r_2 + x_2^0 r_1}{r_1 + r_2}$. So, performing strategic reflexion, the agents "exchange" reputations with one another and report the opponent's opinion.

[11] If each agent adheres to truth-telling, reflexion makes no sense.

[12] Note that such goals of agent's behavior differ from the ones in active expertise models (an agent wants the resulting opinion to match his actual opinion, not the reported message).

[13] A more intellectual player at least expects that the other agents-players may perform reflexion.

The stability condition [168] of the reflexive equilibrium (2.52) is the coincidence of the resulting opinions defined by (2.40) and (2.53):

$$\sum_{i \in N} r_i[s_i^*(r, x_{-i}^0) - x_i^0] = 0. \tag{2.54}$$

Now, we will discuss in brief the case of *informational reflexion*, which precedes strategic reflexion [168]. Denote by Σ the set of all possible finite sequences of indexes from N. Let $r_{i\sigma}$ be the belief of agent i about the reputation of agent σ [168], where $i \in N$ and $\sigma \in \Sigma$. For example, r_{ij} is the belief of agent i about the reputation of agent j, r_{ijk} is the belief of agent i about the belief of agent j about the reputation of agent k, and so on. (Under common knowledge, $r_{ij} = r_j$ for any $i, j \in N$). Such an awareness structure can be considered using the apparatus of reflexive games [168], particularly for *informational equilibrium* design and stability analysis. This represents a topical issue for future investigations.

Concluding Sect. 2.4, note that informational influences for achieving certain agents' awareness about their reputations in a social network is a kind of informational control. Such control, including its particular case—manipulation (see above), is also an interesting field of research.

Example 2.24 Consider an interaction of three agents ($n = 3$) with strategic reflexion. The initial opinions are $x_1^0 = 1, x_2^0 = 2$, and $x_3^0 = 3$. All agents have the same reputation equal to 1. In the case of truth-telling by all agents, the resulting opinion would be $X = 2$.

Using (2.52) find

$$s_1^* = 5/2, \quad s_2^* = 2, \quad s_3^* = 3/2.$$

With these messages, the resulting opinion is $\hat{X} = 2$, i.e., condition (2.54) holds. An example where (2.54) fails is the scenario with $x_3 = 4$. Then

$$s_1^* = 3, \quad s_2^* = 5/2, \quad s_3^* = 3/2, \quad \text{and} \quad \hat{X} = 7/3 > X = 2.$$

Consider an example of informational reflexion that involves two agents ($n = 2$). The initial opinions are $x_1^0 = 1$ and $x_2^0 = 2$; the reputations, $r_1 = 2$ and $r_2 = 1$. The resulting opinion under truth-telling would be $X = 4/3$. In the case of strategic reflexion, $X = 5/3$.

Now, choose the awareness structure $1 \to 2 \leftrightarrow 21$. In other words, agent 2 has the belief $r_{21} = 3$ about the opponent's reputation and considers it as common knowledge. Agent 1 is completely informed of it. Calculate the informational equilibrium: following (2.52), agent 2 chooses $s_2^*(r_{21}, r_2, x_1^0) = x_1^0$ (for two agents, this choice does not depend on the beliefs of agent 2 about the opponent's reputation!), expecting the same message from agent 1. On the other hand, agent 1 chooses his/her best response s_1^* from the condition $\frac{s_1^* r_1 + x_1^0 r_2}{r_1 + r_2} = s_1^*$, i.e., $s_1^* = x_1^0$. The informational equilibrium (x_1^0, x_1^0) is *stable* yet *false*: it drives to the resulting opinion 2/3, which differs from the resulting opinion $X = 4/3$ under complete awareness. •

Further studies of academic and practical interest include the following: (1) generalizations of these models under weaker assumptions, in the first place, incomplete and asymmetric awareness of agents; (2) game-theoretic models of informational control and informational confrontation with uncertainty, reflexion and cooperation of agents.

2.5 Informational Control and Trust of Network Members

This section considers two setups of informational control problems for the trust of social network members. The first setup proceeds from reputation control while the second from mutual trust control of agents.

Reputation control. Assume there exists a set of agents $M \subseteq N$ (agents of influence) whose reputation can be affected by a control subject (Principal).

Let the initial opinions of all agents and also the reputations of all agents except those of influence be given and fixed. Other known parameters of the model include the following: $c_j(r_j)$ as the Principal's cost to establish a reputation r_j for the agent of influence j, where $j \in M(|M| = m)$; $H(X)$ as the Principal's payoff from the resulting opinion X. Denote by $r_M = \left(r_j\right)_{j \in M}$ the reputation vector of all agents of influence and by $C_0(r_M) = \sum_{j \in M} c_j(r_j)$ the total Principal's cost.

In accordance with the results of Sect. 2.4, the resulting opinion of all social network members depends on their initial opinions and reputations, i.e.,

$$X(r_M) = \frac{1}{\sum_{i \in N \setminus M} r_i + \sum_{j \in M} r_j} \left[\sum_{i \in N \setminus M} r_i x_i^0 + \sum_{j \in M} r_j x_j^0 \right]$$

Assume there are no reputation constraints, and define the Principal's goal function as the difference between his/her payoff and cost:

$$\Phi(r_M) = H(X(r_M)) - C_0(r_M).$$

In this case, *reputation control design* can be written as a standard optimization problem of the form

$$H\left(\frac{1}{\sum_{i \in N \setminus M} r_i + \sum_{j \in M} r_j} \left(\sum_{i \in N \setminus M} r_i x_i^0 + \sum_{j \in M} r_j x_j^0 \right)\right) - \sum_{j \in M} c_j(r_j) \to \max_{r_M \geq 0}.$$

Control of trust matrix elements. Further exposition mostly relies on a hypothesis that the object of control is the opinions of agents. Generally speaking, the Principal may affect these opinions at some times. Such informational influences change the resulting opinions of the agents, making them more beneficial to the Principal. However, informational control can be applied to the mutual trust or influence of agents as well. Using proper variations for the degrees of mutual trust (the elements of a trust matrix), the Principal may also achieve his/her goals.

For formal modeling of trust control, recall that at a time $t \geq 0$ the state of a social network without control (i.e., the opinion vector of all agents) satisfies the relationship

$$x^t = (A)^t x, \tag{2.55}$$

where $x = x^0$ is the initial network state and A denotes a direct influence matrix of dimensions $n \times n$. Assume the Principal's *trust control* is implemented in form of an additive variation of the matrix A with a control matrix $V = ||v_{ij}||$. Let this matrix belong to a set of admissible controls $\bar{V}$. The set $\bar{V}$ describes the Principal's capabilities to influence certain mutual connections of agents and also the existing resource constraints.

First, consider the case in which the Principal applies a single control at the initial step. As a result, expression (2.55) takes the form

$$x^t = (A + V)^t x. \tag{2.56}$$

Note that the new influence matrix $(A + V)$ of the system must be also stochastic. This leads to the following constraints on the choice of V:

$$\begin{cases} \forall i \in N: & \sum_{j \in N} v_{ij} = 0, \\ \forall i,j \in N: & -a_{ij} \leq v_{ij} \leq 1 - a_{ij}. \end{cases} \tag{2.57}$$

Denote by $\hat{V}$ the set of all matrices of dimensions $n \times n$ that satisfy conditions (2.57).

Assume the Principal's goal function $\Phi(x^t, V)$—the control efficiency criterion—depends on the opinions of all agents at step t and also on the control matrix. Then the control problem is to choose an admissible control matrix that maximizes the efficiency criterion:

$$\Phi(x^t, V) \to \max_{V \in \bar{V} \cap \hat{V}}.$$

Example 2.25 Get back to the social network of three agents considered in Examples 2.17 and 2.18. Assume at the initial step the Principal may vary (increase or decrease) the degree of trust of agent 2 to agent 3 at most by a given constant Δ, where $\Delta \leq \min\{\alpha, 1 - \alpha\}$. Therefore, the set of admissible controls consists of the matrices $V = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -v & v \\ 0 & 0 & 0 \end{pmatrix}$, where $|v| \leq \Delta$. Let the Principal's goal be the maximal total opinion of all agents at a fixed step t: $\Phi = x_1^t + x_2^t + x_3^t \to \max\limits_{|v| \leq \Delta}$.

Direct calculations yield

$$x^t = (A+V)^t x = \begin{pmatrix} (1-\alpha-v)^{t-1}x_2 + [1-(1-\alpha-v)^{t-1}]x_3 \\ (1-\alpha-v)^t x_2 + [1-(1-\alpha-v)^t]x_3 \\ x_3 \end{pmatrix}.$$

Hence,

$$\begin{aligned} \Phi = x_1^t + x_2^t + x_3^t &= [(2-\alpha-v)(1-\alpha-v)^{t-1}]x_2 \\ &\quad + [3-(2-\alpha-v)(1-\alpha-v)^{t-1}]x_3 \\ &= 3x_3 + (2-\alpha-v)(1-\alpha-v)^{t-1}(x_2 - x_3). \end{aligned}$$

Based on this formula, we may draw the following conclusions.

(a) If $x_2 > x_3$, the Principal's optimal control is $v = -\Delta$;
(b) If $x_2 < x_3$, the Principal's optimal control is $v = \Delta$;
(c) If $x_2 = x_3$, the Principal has no influence on situation (formally speaking, any admissible control is optimal). •

In the general case, the Principal may apply control at different steps, with specific constraints for each step.

For step τ, designate as $\bar{V}^\tau$ the set of admissible controls and as V^τ the control matrix itself. Then the influence matrix at step t is calculated by

$$A^t = A + \sum_{\tau=0}^{t} V^\tau, \tag{2.58}$$

while the recursive formula of network states can be written as

$$x^{t+1} = (A^t + V^t)x^t. \tag{2.59}$$

On a planning horizon T, control matrices $V^\tau, \tau = 0, \ldots, T-1$, are admissible only if all the corresponding matrices $A^t, t = 1, \ldots, T-1$, are stochastic in rows:

$$\begin{cases} \forall i \in N, \forall t \in \{0, \ldots, T-1\}\colon \sum\limits_{j \in N} v_{ij}^t = 0; \\ \forall i,j \in N, \forall t \in \{0, \ldots, T-1\}\colon -a_{ij} \leq \sum\limits_{\tau=0}^{t} v_{ij}^t \leq 1 - a_{ij}. \end{cases} \tag{2.60}$$

Denote by $\hat{V}(T)$ the set of all finite sequences of matrices $(V^0, \ldots, V^{T-1})$ of dimensions $n \times n$ that satisfy conditions (2.60).

With (2.58), relationship (2.56) takes the form

$$x^T = \left(\prod_{t=0}^{T-1} A^{T-t-1}\right)x.$$

Assume the Principal's goal function depends on the resulting opinions of all agents at step T and also on the control matrices at steps 0, …, $T - 1$. Then the control problem is to choose an admissible sequence of control matrices that maximizes the efficiency criterion:

$$\Phi\left(x^T, V^0, \ldots, V^{T-1}\right) \to \max_{\substack{V^0 \in \bar{V}^0, \ldots, V^{T-1} \in \bar{V}^{T-1} \\ (V^0, \ldots, V^{T-1}) \in \hat{V}(T)}}. \tag{2.61}$$

The general control problem (2.61) is rather difficult. So it seems promising to identify and analyze some special cases with good practical interpretations.

Also note another distinctive feature of trust control as follows. If at an initial step the opinions of agents belong to some interval, they will stay there under any control. Let us state this fact rigorously.

Proposition 2.8 Let the Principal apply trust control. Then, for any t = 0, 1, … *and any* $i \in N$,

$$x_{\min} \le x_i^t \le x_{\max}, \tag{2.62}$$

where $x_{\min} = \min\{x_1^0, \ldots, x_n^0\}, \quad x_{\max} = \max\{x_1^0, \ldots, x_n^0\}$.

Proof of Proposition 2.8 Denote by $\tilde{A}^t$ the stochastic matrix figuring in the recursive formula of expression (2.59): $\tilde{A}^t = A^t + V^t$.

Prove by induction on t. The base case: for t = 0, inequality (2.62) holds by the definition of $x_{\min}$ and $x_{\max}$. The inductive step: let (2.62) be true for all $i \in N$ at some step t. Since $\sum_{j \in N} \gamma_{ij}^t = 1$ for any $i \in N$, write the opinion of agent i at step (t + 1) using the elements of the matrix $\tilde{A}^t = \left\|\gamma_{ij}^t\right\|$ as $x_i^{t+1} = \sum_{j \in N} \gamma_{ij}^t x_j^t$.

The right-hand side of this expression satisfies the following chain of inequalities:

$$x_{\min} = \sum_{j \in N} \gamma_{ij}^t x_{\min} \le \sum_{j \in N} \gamma_{ij}^t x_j^t \le \sum_{j \in N} \gamma_{ij}^t x_{\max} = x_{\max}.$$

Hence, $x_{\min} \le x_i^{t+1} \le x_{\max}$. The proof of Proposition 2.8 is complete.

Corollary 2.8.1 *Assume at the initial step the opinions of all agents coincide with each other. If the Principal applies trust control, the opinions still remain the same for all subsequent steps.*

For proving Corollary 2.8.1, just let $x_{\min} = x_{\max}$ under the hypotheses of Proposition 2.8.

As a matter of fact, Proposition 2.8 considerably restricts the Principal's capabilities to achieve his/her goals with trust control: the opinions stay within the range $[x_{\min}, x_{\max}]$ under any controls. However, the following result is also the case.

Proposition 2.9 Assume there are no constraints on the Principal's admissible controls. Then an arbitrary value $x^* \in [x_{\min}, x_{\max}]$ can be implemented as the opinion of each agent using trust control in a single step.

Proof of Proposition 2.9 If $x_{\min} = x_{\max}$, this fact is obvious (see Corollary 2.8.1). Let $x_{\min} < x_{\max}$ and consider a given value $x^* \in [x_{\min}, x_{\max}]$. Find γ from the relationship $x^* = \gamma x_{\min} + (1-\gamma)x_{\max}$, i.e.,

$$\gamma = \frac{x_{\max} - x^*}{x_{\max} - x_{\min}}.$$

Next, let $k \in N$ and $l \in N$ be such that $x_k = x_{\min}$ and $x_l = x_{\max}$. Define the control matrix $V = ||v_{ij}||$ in the following way: for all $i \in N$, $v_{ik} = \gamma - a_{ik}, v_{il} = 1 - \gamma - a_{il}$, and $v_{ij} = -a_{ij}$, where $j \in N \backslash \{k, l\}$.

Then all elements of the column vector $x^1 = (A + V)\, x$ are x^* because

$$\sum_{j \in N} (a_{ij} + v_{ij})x_j = \gamma x_k + (1-\gamma)x_l = x^*.$$

Consequently, the opinion x^* has been implemented as the consensus of all agents in a single step. The proof of Proposition 2.9 is complete.

Concluding this section, we emphasize that an independent analysis of reputation control and trust control is convenient in theoretical terms. In practice, these control approaches should be (and often are) applied jointly. So further investigations will be focused on the development of integrated control models based on the results established in Sects. 2.2 and 2.3 (opinion control) and Sects. 2.1, 2.4 and 2.5 (trust/reputation).

2.6 Informational Control and Network Structure[14]

The case of Principal's complete awareness. Following [45], consider a situation (see the model in Sect. 2.1) in which the agents change their opinions in accordance with the linear law

[14]This section was written jointly with D.N. Fedyanin.

$$x_i^{\tau} = \sum_j a_{ij} x_j^{\tau-1}, \quad i \in N. \tag{2.63}$$

Assume the Principal is interested in the maximal total value of the agents' characteristics $\sum_{i \in N} x_j^{\infty}$. For achieving his/her goal, at the initial step the Principal applies controls u_i, thereby changing the characteristics of all agents. Consequently, the total resulting variation of the agents' characteristics makes up

$$F = \sum_{j \in N} (\mathrm{A}^{\infty} u)_j = \sum_{j \in N} \left(\sum_{i \in N} a_{ij}^{\infty} \right) u_j = \sum_{j \in N} w_j u_j, \tag{2.64}$$

where $u = (u_{1,} \ldots, u_n)$.

The function $F(u)$ in (2.64) is the Principal's utility function subject to maximization. Clearly, under limited control resources (e.g., when just k among n components of the vector u are nonzero), the Principal should affect the agents with higher influence levels. Indeed, this would guarantee greater payoff for the Principal.

In the sequel, the following situation will be studied under different types of the Principal's awareness. The Principal can apply unit control (value 1) to k agents only, where $1 \leq k < n$. The two questions of interest are

(1) What is the optimal control strategy of the Principal?
(2) Which network structures are (most or least) beneficial to the Principal?

With complete awareness about the influence levels of all agents, the Principal should apply control to the k most influential agents (see the discussion above). The Principal's benefit (goal) will be identified with the maximal value of his/her utility function (2.64).

The following intermediate result will be useful for further exposition.

Lemma 1 For any natural number n and any nonnegative real numbers w_i, $i \in N = \{1, \ldots, n\}$, such that $\sum_{i \in N} w_i = n$, there exists a direct influence matrix in which the influence level of agent i is w_i.

Proof of Lemma 1 Choose a natural number n and nonnegative real numbers w_i, $i \in N$, such that $\sum_{i \in N} w_i = n$. Construct the matrix A of the elements

$$a_{ij} = \frac{w_j}{n}.$$

This is a stochastic matrix and hence a direct influence matrix in a certain network. At the same time, this matrix has invariance with respect to multiplication by itself:

$$\sum_{k \in N} a_{ik} a_{kj} = \sum_{k \in N} \frac{w_k}{n} \frac{w_j}{n} = \frac{w_j}{n^2} \sum_{k \in N} w_k = \frac{w_j}{n} = a_{ij}.$$

Therefore, $A^{\infty} = A$, and the influence levels of the agents are

$$\sum_{i \in N} a_{ij} = \sum_{i \in N} \frac{w_j}{n} = \frac{w_j}{n} \sum_{i \in N} 1 = w_j.$$

Thus, the matrix A represents the desired direct influence matrix. •

Now, it is possible to establish a couple of propositions that describe the least and most beneficial networks from the Principal's viewpoint.

Proposition 2.10a *Under the Principal's complete awareness, the most beneficial network to him/her is the one in which the influence levels of at most k agents are nonzero.*

Proof of Proposition 2.10 The maximal value of the Principal's utility function (2.64) is n. Really,

$$F = \sum_{j \in N} w_j u_j \leq \sum_{j \in N} w_j = n.$$

This value is achieved if $u_j = 1$ for all j such that $w_j > 0$. •

On the other hand, the least beneficial network is characterized by

Proposition 2.10b *Under the Principal's complete awareness, there exists a unique least beneficial network to him/her in which the influence levels of all agents coincide with each other:*

$$w_1 = \cdots = w_n = 1. \tag{2.65}$$

Proof of Proposition 2.10b Assume on the contrary that there exists a network that violates condition (2.65), being the least beneficial to the Principal. Without loss of generality, rearrange the agents in the non-ascending order of their influence levels. The resulting network satisfies the inequality $w_1 > w_n$ and hence, for some index $l \in N$,

$$w_1 = \cdots = w_l > w_{l+1} \geq \cdots \geq w_n. \tag{2.66}$$

On the strength of Lemma 1, it is possible to construct a network with the following property: first l agents have smaller influence levels while the rest (from $l + 1$ to n inclusive) higher influence levels. Moreover, this is done without violating expressions (2.63) and (2.66). In such a network, the Principal's goal function takes a smaller value than in the original one. This contradicts the hypothesis that the original network is the least beneficial to the Principal. •

The case of unaware Principal. Now, consider a situation in which the Principal knows nothing about the influence levels of specific agents. So he/she applies the same unit control to k randomly chosen agents from the set N (1 < k < n). As before, the Principal seeks to maximize the sum of the resulting characteristics of all agents.

In the case of the unaware Principal, the controls u_i, $i = 1,\ldots,n$, are random variables, each taking value 0 or 1 so that

$$\sum_{i \in N} u_i = k.$$

As the Principal chooses k random agents among n ones, the probability of the event $u_i = 1$ (the unit control is applied to agent i) makes up k/n. The expected value (mean) of u_i is the same:

$$p(u_i = 1) = \mathrm{E}u_i = \frac{k}{n}.$$

The Principal's utility

$$F = \sum_{j \in N} w_j u_j$$

becomes a random variable too. Using the linear property of mathematical expectation, we obtain the following formula for the expected value of F:

$$\mathrm{E}F = \mathrm{E}\left(\sum_{j \in N} w_j u_j\right) = \sum_{j \in N} w_j \mathrm{E}(u_j) = \frac{k}{n}\sum_{j \in N} w_j = \frac{k}{n} \cdot n = k.$$

Therefore, the Principal's mean payoff is independent of the agents' influence levels. In other words, on the average the Principal obtains the same result irrespective of the mutual influence levels of all agents. Note that this result coincides with the least beneficial one over all possible network structures in the case of complete awareness, see the previous paragraph.

Variance is another important characteristic of the Principal's utility. By standard assumption, a rational control subject strives to minimize the utility variance under uncertain decision conditions. So let the Principal be interested in small utility variance.

Proposition 2.11 *Let the Principal be unaware of the influence levels of all agents. Then the most beneficial network to him/her in terms of minimal utility variance is the one with the same influence levels of all agents. His/her least beneficial network is the one that contains a single agent with a nonzero influence level.*

Proof of Proposition 2.11 Recall that the random variables u_i, $i = 1,\ldots,n$, are pairwise dependent. Hence, the variance DF of the Principal's utility is calculated by (e.g., see the book [191]):

$$\mathrm{D}F = \mathrm{D}\left(\sum_{i\in N} w_i u_i\right) = \sum_{i\in N} w_i^2 \mathrm{D}(u_i) + 2\sum_{i>j} w_i w_j \mathrm{cov}(u_i, u_j). \tag{2.67}$$

By analogy, the variance of the variables u_i, $i = 1,\ldots,n$, takes the form

$$\mathrm{D}u_i = \mathrm{E}(u_i^2) - \mathrm{E}^2(u_i) = \frac{k}{n} - \left(\frac{k}{n}\right)^2.$$

The random product $u_i u_j$ is 1 or 0. Its mean—the probability that the random product is 1—makes up

$$\mathrm{E}(u_i u_j) = p(u_i u_j = 1) = p(u_i = 1)p(u_j = 1|u_i = 1) = \frac{k}{n}\cdot\frac{k-1}{n-1}.$$

Here $p(A|B)$ denotes the conditional probability of A given B. Consequently,

$$\begin{aligned}\mathrm{cov}(u_i, u_j) &= \mathrm{E}\left[(u_i - \mathrm{E}u_i)(u_j - \mathrm{E}u_j)\right] = \mathrm{E}u_i u_j - (\mathrm{E}u_i)^2 \\ &= \frac{k}{n}\cdot\frac{k-1}{n-1} - \left(\frac{k}{n}\right)^2.\end{aligned}$$

Substituting the values $\mathrm{D}u_i$ and $\mathrm{cov}(u_i,u_j)$ into (2.67) yields

$$\begin{aligned}\mathrm{D}F &= \sum_i w_i^2 \cdot \frac{k}{n}\left(1 - \frac{k}{n}\right) + 2\sum_{i>j} w_i w_j \frac{k}{n}\cdot\left(\frac{k-1}{n-1} - \frac{k}{n}\right) \\ &= \frac{k(n-k)}{n^2}\left(\sum_i w_i^2\right) - \frac{2k(n-k)}{n^2(n-1)}\left(\sum_{i>j} w_i w_j\right).\end{aligned}$$

The last expression can be written in compact form using the algebraic formula

$$\sum_{i>j}(w_i - w_j)^2 = (n-1)\sum_{i\in N} w_i^2 - 2\sum_{i>j} w_i w_j. \tag{2.68}$$

The total number of terms in the left-hand side of (2.68) is $n(n-1)/2$. So define Δ as the mean-square difference of the influence levels of noncoinciding agents, i.e.,

$$\Delta = \frac{2}{n(n-1)}\sum_{i>j}(w_i - w_j)^2.$$

With this notation and (2.67), the variance takes the form

$$\mathrm{D}F = \frac{k(n-k)}{2n}\Delta.$$

Evidently, under fixed n and k, the variance DF achieves minimum if $\Delta = 0$, i.e., all agents have the same influence levels.

Without loss of generality, rearrange the agents in the non-ascending order of their influence levels: $w_1 \geq \cdots \geq w_n$.

To find the maximal value of DF, just demonstrate that $\Delta_{\max} = 2n$ is achieved if

$$w_1 = n; \quad w_i = 0, \quad i > 1. \tag{2.69}$$

(In practical interpretation, agent 1 possesses all influence in the network.)

Indeed, we have the relationship

$$n^2 = \left(\sum_{i \in N} w_i\right)^2 = \sum_{i \in N} w_i^2 + 2\sum_{i > j} w_i w_j,$$

which implies

$$\begin{aligned}\sum_{i > j} (w_i - w_j)^2 &= (n-1)\sum_{i \in N} w_i^2 - 2\sum_{i > j} w_i w_j \\ &= (n-1)\left(n^2 - 2\sum_{i > j} w_i w_j\right) - 2\sum_{i > j} w_i w_j \\ &= n^2(n-1) - 2n\sum_{i > j} w_i w_j.\end{aligned}$$

Under condition (2.69), the right-hand side of this formula achieves the maximum $n^2(n–1)$, and

$$\Delta = \frac{2}{n(n-1)}\sum_{i > j} (w_i - w_j)^2 = \frac{2}{n(n-1)} n^2(n-1) = 2n.$$

Thus, the maximal value of DF is $k(n–k)$ corresponds to the network structure with a single agent of nonzero influence level. •

The case of Principal's partial awareness. Finally, we will analyze the situation in which the Principal has partial awareness as follows.

Let the network structure be decomposed into nonintersecting *informational subsets* so that, for each subset, the Principal does not distinguish among agents but knows the number of agents and their total influence level.

Denote these subsets by G_i, $i \in M = \{1, 2, \ldots, m\}$:

$$G_1 \cup \cdots \cup G_m = N.$$

In this case, the Principal's strategy is to choose the impact on each informational subset, i.e., the number k_i of agents from the subset $G_i, i \in M$, to be controlled. Note that in each subset the Principal takes randomly chosen agents. The total number of agents is still equal to k. Of course, k_i does not exceed the number n_i of agents in the informational subset G_i:

$$\sum_{i \in M} k_i = k; \quad k_i \le n_i, \; i \in M.$$

Similar to the case of the unaware Principal, his/her utility is a random variable with the expected value

$$\begin{aligned} \mathrm{E}F &= \mathrm{E}\left(\sum_{j \in N} w_j u_j\right) = \mathrm{E}\left(\sum_{i \in M}\sum_{j \in G_i} w_j u_j\right) = \sum_{i \in M}\sum_{j \in G_i} w_j \mathrm{E}(u_j) \\ &= \sum_{i \in M}\sum_{j \in G_i} w_j \frac{k_i}{n_i} = \sum_{i \in M} k_i \left(\frac{1}{n_i}\sum_{j \in G_i} w_j\right) = \sum_{i \in M} k_i \bar{w}_i, \end{aligned} \tag{2.70}$$

where $\bar{w}_i$ indicates the average influence level of the agents from the subset G_i.

Note that expression (2.70) is analogous to Formula (2.64). It implies that the Principal should affect the subsets with the maximal average influence level. The most beneficial network to the Principal has the following structure.

Proposition 2.12a *Under the Principal's partial awareness, the most beneficial network to him/her is the one in which the total number of agents in informational subsets with nonzero average influence levels does not exceed k.*

Proof of Proposition 2.12 The maximal expected value of the Principal utility (2.70) is

$$\mathrm{E}F = \sum_{i \in M} k_i \bar{w}_i = \sum_{i \in M} \frac{k_i}{n_i} \sum_{j \in G_i} w_j \le \sum_{i \in M}\sum_{j \in G_i} w_j = \sum_{j \in N} w_j = n.$$

This value is achieved if $k_i = n_i$ for all $i \in M$ such that $\bar{w}_i > 0$. •

On the other hand, the least beneficial network is as follows.

Proposition 2.12b *Under the Principal's partial awareness, there exists a unique least beneficial network to him/her in which the average influence levels of all informational subsets coincide*:

$$\bar{w}_1 = \ldots = \bar{w}_m = 1. \tag{2.71}$$

Proof of Proposition 2.12 Without loss of generality, rearrange the subsets in the non-ascending order of their average influence levels:

$$\bar{w}_1 \geq \ldots \geq \bar{w}_m$$

Assume on the contrary that there exists a network that violates condition (2.71), being the least beneficial one to the Principal. This network satisfies the inequality $\bar{w}_1 > \bar{w}_m$ and hence, for some index $l \in M$,

$$\bar{w}_1 = \cdots = \bar{w}_l > \bar{w}_{l+1} \geq \cdots \geq \bar{w}_m. \tag{2.72}$$

In accordance with Lemma 1, it is possible to construct a network with smaller average influence levels of first l informational subsets and larger average influence levels of the rest subsets with indexes from l + 1 and n inclusive. Moreover, this network will satisfy relationships (2.63) and (2.72). The Principal's goal function will take a smaller value on this network in comparison with the original counterpart. This obviously contradicts the fact that the original network is least beneficial to the Principal. •

Actually, Propositions 2.12a and 2.12b are natural extensions of Propositions 2.10a and 2.10b, respectively: informational subsets can be treated as meta agents with set-average influence levels.

A promising line of further research is to study different types of the Principal's awareness about the structural properties of the network and related issues of informational control efficiency.

2.7 Actional Model of Influence

Following the papers [91, 92], this section describes a formal model of actions spreading through a social network and also an associated influence calculation method. In this model, the basic element of analysis is an action performed by an agent (a network user), which explains the term "actional."

Network members are agents belonging to a fixed set

$$N = \{1, 2, \ldots, n\}.$$

They choose actions from a fixed set of admissible action types

$$K = \{1, 2, \ldots, k\}$$

at some times within an interval T. Possible types of actions include writing a post, a comment for a post, etc. Denote by Δ the set of actions (writing a concrete post, comment, and so on). By assumption, this set is finite.

Each action $a \in \Delta$ is characterized by three parameters, namely, the agent performing it, the type of this action and the corresponding time t:

$$a(i,j,t), \quad i \in N, \quad j \in K, \quad t \in T.$$

Define a function $\alpha(a)$ that associates each action $a \in \Delta$ with a corresponding agent $\alpha \in N$ who performed it.

Next, consider a binary partial-order relation of the form "a causes b," which is defined on the action set and designated as

$$a \to b.$$

(Note that it is equivalent to "b is a consequence of a.") For example, in a real online social network, a corresponds to writing a post and b to writing a comment for this post.

Suppose the binary relation satisfies the properties of reflexivity, anti-symmetry, and transitivity.

If $a \to b$ and $a \neq b$ but there exists no $c \in \Delta$ such that $a \to c$ and $c \to b$, we will say that a is a *direct cause* of b (equivalently, b is a *direct consequence* of a). As a result, it is possible to separate a class of binary relations in which each action has at most one direct cause. Such binary relations will be called *unique*.

We give an example of a nonunique binary relation. Let a be a post, b a comment for this post, and c another post with the feature that b contains a reference to c. Then, if $a \to b$ and $c \to b$ are assumed to hold, the binary relation is nonunique.

For a given set $A \subseteq \Delta$, it is possible to define the set of all actions representing the consequences of actions from A:

$$\pi(A) = \{b \in \Delta | \exists a \in A : a \to b\}.$$

Note that, for all sets $A \subseteq \Delta$, the inclusion $A \subseteq \pi(A)$ holds by the reflexivity of the binary relation.

Among all actions Δ, separate the set Δ^0 of *initial actions* that are not the consequences of any other action:

$$\Delta^0 = \{a \in \Delta | \forall b \in \Delta (b \to a) \Rightarrow (a = b)\}.$$

Note that, for unique binary relations, each action has a unique initial action as its cause. Therefore, the sets $\pi(A)$ and $\pi(B)$ do not intersect for any nonintersecting sets A, $B \in \Delta^0$.

Recall that there exist many calculation methods for user influence levels in online social networks. However, generally they ignore why and from whose viewpoint an influence level is estimated. Meanwhile, these issues seem extremely important if influence level is treated as the capability of stimulating others to certain actions.

Therefore, consider the influence level calculation problem from the viewpoint of a control subject (a *Principal*). Being guided by his/her personal interests, the Principal chooses which actions of agents in a social network are significant (they can be desired or undesired for the Principal). To formalize the Principal's viewpoint in influence calculation, introduce *the significance of an action set* as a function $\Phi(S)$ defined by

$$\Phi: 2^{\Delta} \to [0, +\infty).$$

Naturally enough, if a certain action set is supplemented by other actions, then its significance increases (at least, does not decrease). So the significance of an action set (in the sequel, simply significance) represents a monotonically increasing function: if $A \subseteq B$, then $\Phi(A) \le \Phi(B)$. In addition, accept an obvious hypothesis that at least some actions have a positive significance, i.e., $\Phi(\Delta) > 0$.

An important class of significance functions consists of the *additive* functions that satisfy the relationship

$$\Phi(A \cup B) = \Phi(A) + \Phi(B)$$

for any nonintersecting sets $A, B \in \Delta$.

Now, consider an approach to calculate influence within the actional model. We will define the influence of a meta-agent (or a meta-user) representing any nonempty subset of the agent set N. In a real social network, these subsets can be formed in different ways, using initially given individual properties or characteristics of separate agents (e.g., during registration of a new user in the network) or using precalculated parameters (including the ones that depend on mutual relations within the network). We emphasize that a meta-agent is each separate agent $i \in N$ (the singleton $\{i\}$) and also the set of all agents N.

For each meta-agent $I \subseteq N$, define the set $\delta \subseteq \Delta$ of actions performed by him/her/it (i.e., by all agents belonging to the set I) in the form

$$\delta_I = \{a \in \Delta | \alpha(a) \in I\}.$$

and also the set of initial actions performed by him in the form

$$\delta_I^0 = \left\{a \in \Delta^0 | \alpha(a) \in I\right\}.$$

Speaking informally, the notion of influence can be comprehended as follows. The influence level of a meta-agent $I \subseteq N$ on a meta-agent $J \subseteq N$ is high if the activity of agents from the set J is to a large extent conditioned by the activity of agents from the set I. This notion can be formalized in different ways, depending on the specifics of a practical problem under study. In this paper, we proceed from the following hypotheses:

(1) of major interest is the influence of initial actions, i.e., the users who introduce new information to a network (in other cases, it is also possible to consider the efficient disseminators of information from other users);
(2) the influence of the whole network (i.e., the set of all agents in it) on each meta-agent is 1, i.e., the total influence on each meta-agent is normalized.

Under the accepted hypotheses, the influence function of a meta-agent I on a meta-agent J can be defined as

$$\chi(I,J) = \begin{cases} \frac{\Phi(\pi(\delta_I^0) \cap \delta_J)}{\Phi(\delta_J)}, & \Phi(\delta_J) > 0; \\ 0, & \Phi(\delta_J) = 0. \end{cases}$$

In the sequel, we assume that $\Phi(\delta_J) > 0$ for any $J \subseteq N$ (i.e., the agents with zero total significance of actions are eliminated from consideration). Then, obviously

$$\chi(I,J) \leq \frac{\Phi(\pi(\delta_N^0) \cap \delta_J)}{\Phi(\delta_J)} = \chi(N,J) = 1.$$

In an important special case, the meta-agent J coincides with the agent set (i.e., $J = N$) and the influence function therefore reflects the influence of a meta-agent I on the whole network, which will be called the *influence level* of the meta-agent I and denoted by $\varepsilon(I)$:

$$\varepsilon(I) = \chi(I,N) = \frac{\Phi(\pi(\delta_I^0))}{\Phi(\Delta)}.$$

Below we present some properties of the influence function introduced in this way.

Proposition 2.13 *The influence function $\chi(I,J)$ is monotonic in the first argument, i.e., for $I_1 \subseteq I_2$ and any J, we have the inequality $\chi(I_1,J) \leq \chi(I_2,J)$.*

Proof Follows from the chain of relationships

$$\begin{aligned} I_1 \subseteq I_2 &\Rightarrow \delta_{I_1}^0 \subseteq \delta_{I_2}^0 \Rightarrow \pi\left(\delta_{I_1}^0\right) \subseteq \pi\left(\delta_{I_2}^0\right) \\ &\Rightarrow \pi\left(\delta_{I_1}^0\right) \cap \delta_J \subseteq \pi\left(\delta_{I_2}^0\right) \cap \delta_J \\ &\Rightarrow \Phi\left(\pi\left(\delta_{I_1}^0\right) \cap \delta_J\right) \leq \Phi\left(\pi\left(\delta_{I_2}^0\right) \cap \delta_J\right) \\ &\Rightarrow \chi(I_1,J) \leq \chi(I_2,J). \quad \bullet \end{aligned}$$

Proposition 2.13 means that, the "larger" is a meta-agent (i.e., the more agents it includes), the higher is his/her/its influence regardless of other circumstances.

Proposition 2.14 *If the binary relation is unique and the significance function is additive, then the influence function is additive in the first argument, i.e., for any $I_1, I_2, J \subseteq N$ such that $I_1 \cap I_2 = \varnothing$, we have the equality*

$$\chi(I_1 \cup I_2, J) = \chi(I_1,J) + \chi(I_2,J).$$

Proof Follows from the chain of relationships

$$\chi(I_1 \cup I_2, J) = \frac{\Phi\left(\pi\left(\delta^0_{I_1 \cup I_2}\right) \cap \delta_J\right)}{\Phi(\delta_J)} = \frac{\Phi\left(\left(\pi\left(\delta^0_{I_1}\right) \cup \pi\left(\delta^0_{I_2}\right)\right) \cap \delta_J\right)}{\Phi(\delta_J)}$$
$$= \frac{\Phi\left(\left(\pi\left(\delta^0_{I_1}\right) \cap \delta_J\right) \cup \left(\pi\left(\delta^0_{I_2}\right) \cap \delta_J\right)\right)}{\Phi(\delta_J)} \stackrel{(*)}{=}$$
$$= \frac{\Phi\left(\pi\left(\delta^0_{I_1}\right) \cap \delta_J\right)}{\Phi(\delta_J)} + \frac{\Phi\left(\pi\left(\delta^0_{I_2}\right) \cap \delta_J\right)}{\Phi(\delta_J)} = \chi(I_1, J) + \chi(I_2, J).$$

Here a key role is played by the equality (*), which appears from the additive property of the function Φ together with the fact that the sets $\pi\left(\delta^0_{I_1}\right)$ and $\pi\left(\delta^0_{I_2}\right)$ (and hence the sets $\pi\left(\delta^0_{I_1}\right) \cap \delta_J$ and $\pi\left(\delta^0_{I_2}\right) \cap \delta_J$) do not intersect due to the uniqueness of the binary relation. •

As easily seen, in this case the influence level of a meta-agent is also an additive function: the equality $\varepsilon(I_1 \cup I_2) = \varepsilon(I_1) + \varepsilon(I_2)$ holds for any nonintersecting sets $I_1, I_2 \subseteq N$.

Some examples of influence level calculation for real online social networks can be found in the papers [91] (Facebook) and [92] (VKontakte).

Consider an example of influence level calculation for *Vkontakte*, a popular online social network in the RuNet (vk.com).[15] Let the posts containing the "Nazarbaev[16]" keyword (in any case-form), their reposts as well as the comments and likes for them be significant from the Principal's viewpoint. As the time interval *T*, choose year 2015—from 0 h 0 min 0 s January 01, 2015, to 23 h 59 min 59 s December 31, 2015).

For this set-up, it suffices to consider the following types of actions: (1) writing an original post or making a repost; (2) writing a comment for a post; (3) putting a like for a post; (4) putting a like for a comment. Hence, the set *K* consists of four elements: *K* = {1, 2, 3, 4}.

Assume the binary relation of causality $a \to b$ holds in the following cases: *a*—writing a post, *b*—writing a comment for this post; *a*—writing a post or a comment, *b*—putting a like; *a*—writing a post, *b*—making its repost. Besides, the relation of causality holds if *a* and *b* coincide.

Under the described conditions, each action is assessed independently, and therefore the significance of an action set $S \subseteq \Delta$ depends additively on each action from this set:

$$\Phi(S) = \sum_{a \in S} \Phi(a).$$

[15]The anonymized data for this study were provided by DSS Lab (dss-lab.ru).

[16]The surname of the President of the Republic of Kazakhstan.

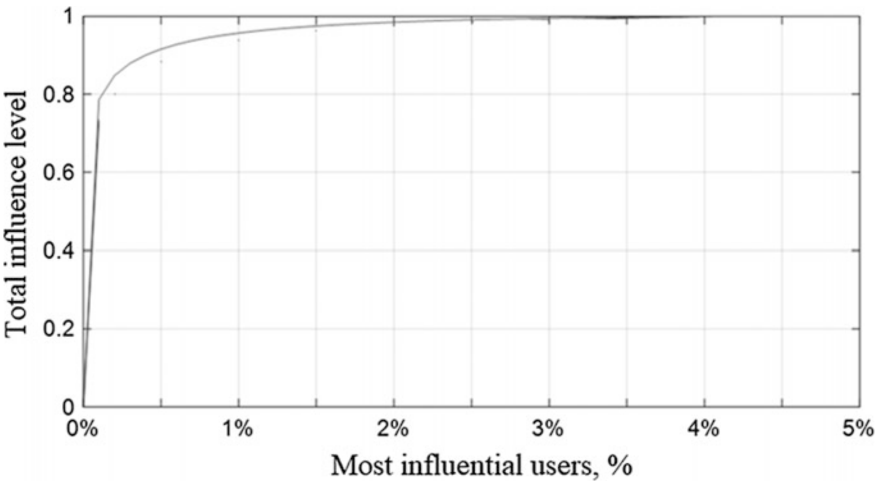

Fig. 2.32 Share of total influence level as function of percentage of most influential users

Let $\Phi(a) = 1/\left|\delta_{\alpha(a)}\right|$, where $|\cdot|$ denotes set cardinality, if a is a post mentioning the keyword that was written within the time interval T, or a comment for such a post that was written within T, or a like for such a post or comment that was put within T; otherwise, let $\Phi(a) = 0$. This definition means that the total significance of the actions of each agent is 1 (in other words, each user has the same significance from the Principal's viewpoint).

Consequently, we have described all necessary data to calculate the influence levels of users. Here are some calculations for separate users (i.e., for singletons I).

As it turned out, the total influence level of 1% (!) of most influential users makes up 94–96% of the total influence level of all users; the total influence level of 2% of most influential users makes up 98% of the total influence level of all users; and the total influence level of 5% of most influential users makes up even 100% of the total influence level of all users. The graph in Fig. 2.32 demonstrates the share of total influence level as a function of the percentage of most influential users.

Therefore, the influence level calculation method suggested in this section allows to identify a small set of users with a highest influence on the actions of other users in an online social network from the Principal's viewpoint (topics of interest and preferences).

Finally, note that the actional model is a family of methods covering various aspects of influence rather than a fixed influence calculation procedure. For example, if users have different significance for the Principal, then this feature can be considered through an appropriate modification of the influence function.

Chapter 3
Models of Informational Confrontation in Social Networks

The paper [161] identified five levels for the description and analysis of active network structures (social networks, mobs, etc.) as follows. At the first (lowest) level, a network is considered in toto; such a description gives no details, yet being essential for a fast analysis of the general properties of these objects. At the second level, the structural properties of a network are examined using the framework of graph theory. Next, at the third level, the informational interaction of agents is analyzed, with a wide range of applicable models (Markovian models, finite-state automata, diffusion of innovations, infection models, to name a few). At the fourth level, informational control problems are formulated and solved. Finally, at the fifth level, informational confrontation is considered as the interaction of subjects affecting a social network for their individual goals. A model adopted at each level of this hierarchy takes into account the outcomes of the previous levels. Therefore, a prerequisite for passing to a next level is the existence of fairly simple interconnected models at the previous levels that are adequate to a modeled reality.

Informational confrontation, which is described at the fifth level, needs simple results on the informational interaction processes and informational control of agents. The first class of models with a complete chain between the lower and upper levels is the models of social networks in terms of consensus problems (or the so-called Markovian models). In Sects. 3.1 and 3.2 of this book, we introduce the corresponding game-theoretic models of informational confrontation (also see the models [93]).

Section 3.1 is dedicated to the models of *distributed informational control* for social network members. The subjects exerting an informational influence on agents often have noncoinciding interests, and the conditions of interests coordination are established for them.

Section 3.2 deals with the model of *informational confrontation* between two control subjects with noncoinciding interests (defender and attacker). *Informational epidemic*, a phenomenon of opinion spreading from one active agent to another passive agent in a network, as well as protection against it are studied. This informational confrontation problem is reduced to a bimatrix game. For a special

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case of social networks described by complete communication graphs, the resulting bimatrix game has at least one Nash equilibrium, as proved below. Also we demonstrate that strategic reflexion in the bimatrix game decreases the number of Nash equilibria (at most two) and may improve the payoffs of players.

The second fruitful example is the game-theoretic modeling of informational contagion "superstructed" over the threshold models of a mob, the approach cultivated in Sect. 3.3. The model suggested in [27] treats a mob as a set of agents with conformity behavior [87]: making their binary choice (being active or passive), such agents rely on the decisions of other agents. The authors [29] introduces the stochastic models of mob control in which a given share of agents is randomly "excited" (made active) and "immunized" (made passive). Assume such controls are applied by different subjects with noncoinciding individual preferences on the "equilibrium" state of a mob. In this case, the subjects get involved into informational confrontation described by a game-theoretic setup.

Note that the game-theoretic models of informational confrontation over active networks have several applications, namely, the informational safety of online social networks, the counteraction to a destructive informational influence on social groups of different scale, the prevention of their massive illegal actions, and others (the details can be found in this book and also in [27, 168]).

3.1 Informational Confrontation: Distributed Control and Interests Coordination

Game-theoretic model of informational confrontation. General setup. Consider a set of *players* who can influence the initial opinions of agents. Each player is interested in a certain resulting opinion of all agents. In this model the agents are passive: they change opinions in accordance with a linear law taking into account the opinions of other agents. In contrast to the agents, the players are active and have individual goals; by choosing their actions, the players affect the agents.[1] Let us describe the game of players.

Introduce the following notations: $M = \{1, 2, \ldots, m\}$ as the set of players; $u_{ij} \in U_{ij} = [-r_{ij}, R_{ij}]$ as the action of player j that is intended to change the opinion of agent i, where $r_{ij}, R_{ij} \geq 0$; $i \in N, j \in M$; $u = (u_1, u_2, \ldots, u_n)$, $u_i = \sum_{j \in M} u_{ij}$, as the influence (control) vector; $\boldsymbol{u} = \|u_{ij}\|$, $\boldsymbol{u}_j = (u_{1j}, u_{2j}, \ldots, u_{nj}) \in U_j = \prod_{i \in N} U_{ij}$; finally, $g_j(X)$: $\mathfrak{R}^n \to \mathfrak{R}^1$ as the goal function of player j.

Assume the influences of players on the opinion of each agent are additive. Then the resulting opinion of agent i makes up

[1]Within this approach, it is possible that some agents coincide with players, acting as control subjects and controlled systems.

$$X_i(\boldsymbol{u}) = \sum_{j \in N} A_{ij}^{\infty} \left[x_j^0 + \sum_{k \in M} u_{jk} \right] = \sum_{j \in N} A_{ij}^{\infty} x_j^0 + \sum_{j \in N} A_{ij}^{\infty} \sum_{k \in M} u_{jk}, \ i \in N. \tag{3.1}$$

In the general case, each player can influence the opinions of all agents (if not, the lower and upper of the corresponding intervals U_{ij} are set equal to 0).

Denoting $G_j(\boldsymbol{u}) = g_j(X_1(\boldsymbol{u}), X_2(\boldsymbol{u}), \ldots, X_n(\boldsymbol{u}))$, $j \in M$, and supposing that the players choose their actions one-time, simultaneously and independently, we naturally arrive at a normal-form game $\Gamma = (M, \{U_j\}_{j \in M}, \{G_j(\cdot)\}_{j \in M})$. This game is defined by specifying the set of players, their admissible actions and goal functions [80]. Using a normal-form game, it is possible to find equilibria, to design cooperative, repeated and other types of games (see the classification in the book [80]).

Example 3.1 The players have the linear goal functions $g_j(X) = \sum_{i \in N} \beta_{ji} x_i^0, j \in M$. For the resulting opinions (3.1), these functions take the form

$$G_l(\boldsymbol{u}) = \sum_{i \in N} \beta_{li} \sum_{j \in M} A_{ij}^{\infty} x_j^0 + \sum_{i \in N} \beta_{li} \sum_{j \in N} A_{ij}^{\infty} \sum_{k \in M} u_{jk}, \ l \in M. \tag{3.2}$$

The actions chosen by the players appear in the second term only. Denote $\gamma_{lj} = \sum_{i \in N} \beta_{li} A_{ij}^{\infty}$, $l \in M$. Since the goal functions are linear in the actions of players, the game under consideration has a dominant strategy equilibrium $\boldsymbol{u}^d$ [80] in which player l chooses an action maximizing $\sum_{j \in M} \gamma_{lj} u_{jl}$ regardless of the actions of all other players:

$$u_{jl}^d = \begin{cases} -r_{jl} & \text{if } \gamma_{lj} < 0, \\ R_{jl} & \text{if } \gamma_{lj} \geq 0, \end{cases} \ j \in N, \ l \in M. \tag{3.3}$$

In other words, each player applies the maximal possible influence to each agent, and the sign of this influence depends on the resulting variations in the opinion of this agent (the values of these variations for the players are defined by the weights $\{\gamma_{lj}\}$). •

Example 3.2 Two players pursue noncoinciding interests. Renumber the agents so that player 1 can influence the initial opinion of agent 1 and player 2 the initial opinion of agent 2. Denote these additive influences by $u_1 \in U_1$ and $u_2 \in U_2$, respectively.

Then the resulting opinions of the agents have the form

$$X_i(u_1, u_2) = \sum_{j \in N} A_{ij}^{\infty} x_j^0 + A_{i1}^{\infty} u_1 + A_{i2}^{\infty} u_2, i \in N. \tag{3.4}$$

Let $\boldsymbol{X}(u_1, u_2)$ be the opinion vector of all agents that consists of components (3.4). The Nash equilibrium $\left(u_1^*, u_2^*\right)$ satisfies the conditions

$$\begin{aligned} &\forall u_1 \in U_1 : g_1\big(X\big(u_1^*, u_2^*\big)\big) \geq g_1\big(X\big(u_1, u_2^*\big)\big); \\ &\forall u_2 \in U_2 : g_2\big(X\big(u_1^*, u_2^*\big)\big) \geq g_2\big(X\big(u_1^*, u_2\big)\big). \end{aligned} \tag{3.5}$$

The resulting opinions of all agents have a rather simple additive dependence on the controls (actions of players), see relationship (3.4). So, based on this model, it is possible to consider games with a fixed sequence of moves (hierarchical games) [71, 80], which are often interpreted as defense–attack games.

The model of Example 3.2 can be easily extended to the case in which each player can influence the initial opinions of an arbitrary set of agents. •

Example 3.3 Each of two players can influence the initial opinion of one agent from a corresponding set $N_1 \subseteq N$ or $N_2 \subseteq N$, respectively, where $N_1 \cap N_2 = \varnothing$. Then the players have to choose an appropriate agent for influence. In this case, the sets of admissible actions are finite. After calculation of the corresponding payoffs, we obtain a bimatrix game and find in analytic form an equilibrium in the class of pure and/or mixed strategies. •

Note that the game-theoretic model of informational confrontation under study proceeds from the assumption that the players are choosing their actions one-time, simultaneously and independently of each other, thereby playing a *normal-form game* [80]. This assumption remains in force for the distributed control model described below.

Distributed control. In many real systems, a given agent is simultaneously subordinated to several control subjects—*Principals*—at the same or different hierarchical levels. The first case is called *distributed control* (also known as *agency* in contract theory [19, 144]) while the second *interlevel interaction* [169]. A prime example of distributed control is *matrix control structures* [165].

This paragraph considers distributed control in social networks: generally, the subjects exerting an informational influence on network members have noncoinciding interests.

A distributed control system (DCS) that includes k control subjects (*Principals*) and a single controlled subject (*agent*) can be described by the schematic diagram in Fig. 3.1.

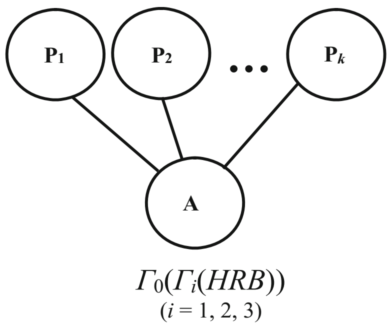

Fig. 3.1 Structure of distributed control system

In a DCS, the Principals controlling the agent are involved in a "game," with a complicated equilibrium. (In Fig. 3.1, this normal-form game is denoted by Γ_0; it is played over a set of hierarchical games (Γ_1, Γ_2 or Γ_3 [71]) under the hypothesis of rational behavior (HRB) accepted for the agent [80]). Particularly, there exist two stable modes of interaction among the Principals—*cooperation* and *competition* [169].

In the cooperation mode, the Principals act jointly for achieving required results of the agent's activity with minimal resources.

In the competition mode, which occurs if the Principals' goals differ appreciably, the resources are spent inefficiently.

Following [165], consider an elementary (basic) model of a DCS and then use it for the distributed control of social networks. Assume an organizational system (OS) consists of a single agent and k Principals. The agent's strategy is to choose an *action* $y \in A$, which incurs costs $c(y)$. As the result of the agent's activity, each Principal i gains some *income* described by a function $H_i(y)$, where $i \in K = \{1, 2, \ldots, k\}$ and K denotes the set of all Principals. Moreover, each Principal i also bears costs $\sigma_i(y)$ to change the opinions and/or actions of the agent (i.e., the incentive paid to the agent). Thus, the goal function of Principal i has the form

$$\Phi_i(\sigma_i(\cdot), y) = H_i(y) - \sigma_i(y),\ i \in K. \tag{3.6}$$

The agent's goal function is given by

$$f(\{\sigma_i(\cdot)\}) = \sum_{i \in K} \sigma_i(y) - c(y). \tag{3.7}$$

The sequence of moves is as follows. The Principals simultaneously and independently choose what opinions of the agent are desired for them (thereby determining their cost functions). Then the agent chooses his/her action. For the Principals' game, further analysis will be confined to the set of *Pareto-optimal* Nash equilibria. In this case [119, 169], the Principals' strategies are

$$\sigma_i(x, y) = \begin{cases} \lambda_i, & y = x \\ 0, & y \neq x \end{cases},\ i \in K. \tag{3.8}$$

This means that the Principals agree to stimulate the agent's choice of a specific action $x \in A$ (*plan*) as well as to share their total costs in a Pareto optimal way, see below. This mode of interaction among the Principals is said to be cooperation.

The conditions of Pareto optimality imply that the total costs of all Principals under plan fulfillment are equal to the agent's costs:

$$\sum_{i \in K} \lambda_i = c(x). \tag{3.9}$$

(As a matter of fact, this is the principle of costs compensation [165] extended to distributed control systems).

For each Principal, the condition of beneficial cooperation may be stated as follows. In the cooperation mode, each Principal gains the utility not smaller than as if he/she motivated the agent independently (by compensating the agent's costs to choose the most beneficial action for this Principal). The utility of Principal i from an "independent" interaction with the agent is defined by [165]

$$W_i = \max_{y \in A} [H_i(y) - c(y)], \ i \in K. \tag{3.10}$$

Construct the vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and denote by

$$S = \left\{ x \in A | \exists \ \lambda \in \Re^k_+ \ : H_i(x) - \lambda_i \geq W_i, \ i \in K, \ \sum_{i \in K} \lambda_i = c(x) \right\} \tag{3.11}$$

the set of such agent's actions that can be implemented by the cooperation of all Principals in a beneficial way. The set of pairs $x \in S$ and corresponding vectors λ is called *the domain of compromise*:

$$\Lambda = \left\{ x \in A, \ \lambda \in \Re^k_+ \, | H_i(x) - \lambda_i \geq W_i, \ i \in K, \ \sum_{i \in K} \lambda_i = c(x) \right\}. \tag{3.12}$$

By definition the cooperation mode takes place if the domain of compromise is non-empty: $\Lambda \neq \varnothing$. In the cooperation mode the agent obtains zero utility. Denote

$$W_0 = \max_{y \in A} \left[\sum_{i \in K} H_i(y) - c(y) \right]. \tag{3.13}$$

The main result of DCS analysis is as follows: the domain of compromise if non-empty if and only if [169]:

$$W_0 \geq \sum_{i \in K} W_i. \tag{3.14}$$

Thus, condition (3.14) expresses an implementability criterion of the cooperation mode. In a practical interpretation, with joint actions the Principals may gain a greater total efficiency in comparison with their individual behavioral strategies. The difference $W_0 - \sum_{i \in K} W_i$ can be treated as a measure of interests' coordination among the Principals and also as a rate of OS emergence.

If condition (3.14) fails ($\Lambda = \varnothing$), the Principals are interacting in the competition mode characterized by *the auction solution*. Let the Principals be rearranged in the ascending order of the values $\{W_i\}$: $W_1 \geq W_2 \geq \cdots \geq W_k$. The winner is the first

Principal who compensates the agent's costs and also offers him/her a utility greater than W_2 by an arbitrarily small value.

General procedure for interests coordination in distributed control systems. This procedure includes the following steps.

(1) Describe the staff and structure of a system that consists of at least several control subjects and one or several controlled agents at lower hierarchical levels.

(2) Define the sequence of moves as follows: the Principals choose controls simultaneously and independently of each other and report them to the agents; then the latter choose actions under these controls, also simultaneously and independently of each other.

(3) Introduce the goal functions and admissible action sets of all system participants. By assumption, the goal function of each agent is additive in the Principals' controls while the goal function of each Principal is additive in the controls reported by him/her to different agents.

(4) Prove that, for the Principals playing their game with Pareto-optimal Nash equilibria, it suffices to consider the quasi-compensatory strategies (3.8), which decompose the interaction of agents in multiagent systems [165]. To this effect, take advantage of the general results established in [119]: for any Pareto-optimal strategy of any Principal, there exists another strategy of at least the same efficiency with nonzero costs of this Principal at most at k points.

Therefore, the problem to find a collection of functions is reduced to the calculation[2] of $k + 1$ parameters—the same incentive-compatible plan for all Principals and the costs of each Principal k.

(5) Write the balance condition (3.9), stating that the total costs of all Principals are precisely compensating the agent's costs under plan fulfilment.

(6) For each Principal, calculate payoff (3.10) from his/her individual interaction with the agent.

(7) Construct the domain of compromise (3.12).

(8) Calculate the maximal possible value of the total payoff of all Principals from their joint activity, see (3.13).

(9) Verify condition (3.14), which guarantees that the domain of compromise is non-empty.

(10.1) If condition (3.14) holds, then cooperation is possible, and the problem is reduced to compromise design: choose a specific point within the domain of compromise.

(10.2) If condition (3.14) fails, then the Principals are interacting in the competition mode, and the game has the auction solution. If the efficiency of this solution is inacceptable, then analyze the feasibility of interests coordination

[2]If the Principals are controlling several ($n \geq 2$) agents, then the number of desired parameters makes up $n (k + 1)$.

for the Principals using higher-level control subjects or the concept of bounded rationality.

This procedure of interests coordination in distributed control systems is rather general. We will employ it for informational control design in social networks.

Within the framework of the model described in Sect. 2.1, denote by $N = \{1, 2, \ldots, n\}$ the set of agents belonging to a social network (members). The agents influence each other, and the degrees of such influences are defined by their reputations or trust. At an initial step, each agent has an *opinion* on some issue. The initial opinions of all network members are described by a column vector y^0 of dimension n that consists of nonnegative initial opinions. The agents are interacting over the social network by exchanging their opinions. The opinion of agent i at step τ is calculated by

$$y_i^\tau = \sum_{j \in N} \alpha_{ij} y_j^{\tau-1}. \tag{3.15}$$

Assume the opinions of all agents converge to a resulting opinion vector $Y = \lim_{\tau \to \infty} y^\tau$ after very many interactions. Then

$$Y = A^\infty y^0. \tag{3.16}$$

Consequently, the resulting opinion vector of all social network members is uniquely defined by their initial opinion vector and the influence/trust matrix. This fact can be used to formulate and solve informational control problems: find purposeful influences on the initial opinions of agents that guarantee required resulting opinions. In the next paragraph, we will consider the problem of interests coordination for the subjects performing informational control.

Conditions of interests coordination for control subjects. Introduce the following notations:

- $\{N_i\}_{i \in K}$ as the aggregate of all subsets of the set N, where N_i indicates the set of agents for the informational influence of Principal i, where $i \in K$;
- $K_j = \{k \in K | j \in N_k\}$ as the set of all Principals with informational influences on agent j, where $j \in N$;
- $c_i(y^0, x)$ as the costs of agent i to change his/her opinion from y_i^0 to x_i; generally speaking, these costs can be a function of the opinion vectors of all agents—the vector y^0 (the initial opinions before informational influences) and the vector x (the initial opinions after informational influences), where $i \in N$;
- $H_i(x)$ as the preferences of Principal i over the set of agents' opinions[3], where $i \in K$;

[3]Of course, it would be natural to assume that the Principals' preferences are defined over the set of the resulting opinions of all agents. But, in accordance with (3.11), the resulting opinions are uniquely determined by the initial ones.

- $\sigma_{ij}(y^0, x)$ as the costs of Principal i to exert informational influences on agent j, where $j \in N_i$ and $i \in K$.

In a practical interpretation, Principals are applying informational controls to agents, thereby changing their opinions. Note that the same agent can be influenced by several Principals simultaneously (a distributed control system). Each Principal tries to change the opinion of some agent for his/her own benefit. So an appropriate model is required for the variations of agent's opinions under such conflicting impacts. Unfortunately, there exist no adequate formal models of this kind to date. In this paragraph, we will establish the conditions of interests coordination for control subjects and give answers to the following questions. In which situations can the Principals agree about their actions? What opinions of the agents must be formed by the Principals? (For each agent, the controls of all Principals must be "consistent" in the sense of no contradictions).

The goal function of Principal i has the form

$$\Phi_i\left(\left\{\sigma_{ij}(\cdot)\right\}_{j\in N_i},\ y^0, x\right) = H_i(x) - \sum_{j\in N_i} \sigma_{ij}(y^0, x),\ i \in K. \tag{3.17}$$

The goal function of agent j is given by

$$f\left(\left\{\sigma_{ij}(\cdot)\right\}_{i\in K_j}, y\right) = \sum_{i\in K_j} \sigma_{ij}(y^0, x) - c_i(y^0, x). \tag{3.18}$$

The sequence of moves is as follows. The Principals simultaneously and independently choose their informational influences (controls) and report them to the agents. As before, further exposition will consider the class of Pareto-optimal Nash equilibria, i.e., the Principals' strategies have the form

$$\sigma_{ij}(y^0, x) = \begin{cases} \lambda_{ij}, & y_j = x_j \\ 0, & y_j \neq x_j \end{cases},\ j \in N_i,\ i \in K. \tag{3.19}$$

So the Principals agree to cooperate, i.e., to form jointly the same opinion vector x as well as to share the associated costs.

The conditions of Pareto optimality imply that the total costs of all Principals are equal to the agent's costs:

$$c_i(y^0, x) = \sum_{j\in K_i} \lambda_{ji},\ i \in N. \tag{3.20}$$

In accordance with condition (3.20), the Principals must distribute the costs to change the opinion of each agent.

By analogy with expression (3.10), calculate

$$W_i = \max_x \left[H_i\left(x_{N_i}, y^0_{-N_i}\right) - \sum_{j \in N_i} c_i(y^0, x) \right], \; i \in K \tag{3.21}$$

and

$$W_0 = \max_x \left[\sum_{i \in K} H_i(x) - \sum_{j \in N} c_j(y^0, x) \right]. \tag{3.22}$$

Construct the matrix $\lambda = \|\lambda_{ij}\|$, and denote by

$$S = \left\{ x \in \Re^n_+ \,|\, \exists\, \lambda \in \Re^{nk}_+ : H_i(x) - \sum_{j \in N_i} \lambda_{ij} \geq W_i, \; i \in K, \; \tilde{n}_i(y^0, x) = \sum_{j \in K_i} \lambda_{ij}, \; i \in N \right\} \tag{3.23}$$

the set of such opinion vectors of agents that can be implemented by the cooperation of all Principals in a beneficial way. The set of all pairs composed of vectors $x \in S$ and corresponding cost matrices λ is called *the domain of compromise* in the distributed control problem of the social network:

$$\Lambda = \left\{ x \in \Re^n_+ \;, \exists\, \lambda \in \Re^{nk}_+ \left| H_i(x) - \sum_{j \in N_i} \lambda_{ij} \geq W_i, \; i \in K, \; c_i(y^0, x) = \sum_{j \in K_i} \lambda_{ij}, \; i \in N \right. \right\}. \tag{3.24}$$

By definition the cooperation mode (*informational cooperation* in social networks) takes place if the domain of compromise (3.24) is non-empty: $\Lambda \neq \varnothing$.

The next result can be established by analogy with the criteria of the non-empty domains of compromise, see [119, 165, 169].

Proposition 3.1 The interests of the control subjects exerting informational influences on social network members can be coordinated if and only if

$$\max_x \left[\sum_{i \in K} H_i(x) - \sum_{j \in N} c_j(y^0, x) \right] \geq \sum_{i \in K} \max_x \left[H_i(x_{N_i}, y^0_{-N_i}) - \sum_{j \in N_i} c_i(y^0, x) \right]. \tag{3.25}$$

Condition (3.25) guarantees the feasibility of interests coordination for control subjects. If this condition fails, then the competition mode is the case. Assume the informational influences of all Principals have no "interference" (i.e., an agent will accept the opinion of the Principal offering the maximal incentive, ignoring the information from other Principals). Then the Principals' game has the auction solution. In a practical interpretation, competition well matches *information warfare* in which the winner is the Principal with the maximal resource (3.21).

Denote

$$x^i = \arg\max_x \left[H_i(x_{N_i}, y^0_{-N_i}) - \sum_{j \in N_i} c_i(y^0, x) \right],\ i \in K.$$

Rearrange the Principals in the descending order of the values $\{W_i\}$: $W_1 \geq W_2 \geq \cdots \geq W_k$. The following fact can be proved for the auction solution using the same approach as in [169].

Proposition 3.2 If condition (3.25) fails, then the resulting opinion of the social network members under informational influences is $(x_{N_1}, y^0_{-N_1})$

Using distributed control models of social networks, we may formulate and solve higher-level problems, e.g., *the division of the spheres of influence*: find which subsets of social network members should be controlled by a certain control subject. A thorough study of the corresponding cooperative game-theoretic models seems an interesting topic for further research.

Moreover, within the framework of the current models all agents in social networks are passive and unintellectual. So the consideration of agents with complex internal structure (in the first place, using *logical models*) that describes their capability for nontrivial *goal implementation*, *goal-setting*, *adaptation* and *reflexion*, is also quite promising. For example, the ideal aim is to reach the general hierarchical architecture of an agent (see Fig. 3.2) that includes the following levels (in the ascending order of their complexity):

(1) *operational level* (*execution level*), which implements behavioral control algorithms under given particular goals and methods for achieving them;
(2) *tactical level*, which implements the algorithms of situation identification and proper behavior choice;
(3) *strategic level*, which implements decision algorithms for particular goals (the cooperative allocation of tasks and functions among different agents in a group, etc.) as well as the algorithms of adaptation, learning, and reflexion[4];
(4) *conceptual level*, which implements goal-setting through choice algorithms for global goals and methods for achieving them (the mechanisms of functioning).

The first step towards active and intellectual agents can be the separation of their opinions (awareness) and actions chosen independently based on the opinions (the relationship between the awareness and actions of agents can be established, e.g. like in decision problems [80, 165] and/or informational control problems [168]).

Another topical issue is to explore control problems over *multinetworks*, with inhomogeneous controls applied to agents within different networks (e.g., one network is used to destabilize the current situation in the eyes of agents; the other to

[4]This level may include four sublevels associated with decision-making, adaptation, learning, and reflexion.

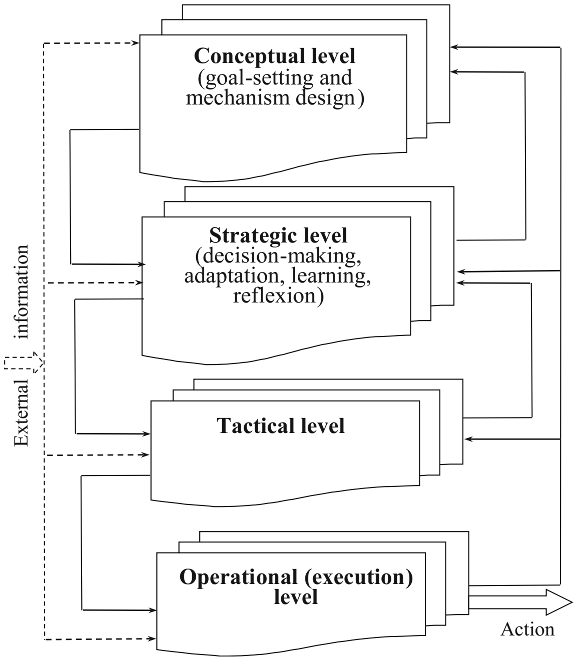

Fig. 3.2 General hierarchical architecture of an agent

report definite information to them; and the third to stimulate their actions). A complete treatment of reflexion—the design and analysis of *phantom networks*—can be fruitful here.

3.2 Informational Epidemic and Protection Against It

In social networks modeling, it is often necessary to consider the opinion spreading over a social network from active agents to passive ones. Such processes are studied, e.g., in the diffusion of innovations, see Chap. 1 of the book. In many applications (e.g., *informational safety*), the cascades of opinion spreading must be detected as soon as possible. In this case, two control subjects are identified that have noncoinciding interests (Attacker and Defender) as well as controlled objects-(network nodes). For each subject, an object possesses some value. Defender chooses a scanning period to monitor the states of all nodes in a given network while Attacker chooses a node for informational attack. This setup naturally leads to informational confrontation, and it is required to solve the game of the control subjects (the Principals' game), i.e., to calculate their equilibrium actions. In Sect. 3.2, we will formulate the informational confrontation problem in a social network as well as describe an algorithm for reducing the Defender–Attacker confrontation to a bimatrix game. As proved below, the Principals' game always has a pure strategy Nash equilibrium if the network graph is complete.

Initial data and assumptions. Agents in a social network have connections defined by a symmetric square matrix $G = (g_{km})_{k,\, m \in N}$. If there exists a connection between agents k and m, then $g_{km} = 1$ (a nonzero degree of trust); otherwise, $g_{km} = 0$. Note that $g_{mm} = 1$ for all m.

In addition to the agents, the model includes two players—A and B. Player B seeks to "infect" the network, i.e., to spread some information (opinion, etc.) over it. To this effect, he/she chooses a certain agent to be infected so that the infection spreads through the whole network. Infection spreading will be modeled using a simple approach as follows. Assume at each discrete time (step) infection covers each agent having connection with another agent infected at the previous step.

Here is the formal description of the model. Consider a sequence of steps $\tau = 0, 1, \ldots$. Denote by $S_\tau \subset N$ the set of infected agents at step τ. Then, at the next step $\tau + 1$, the set of infected agents include those infected earlier and also those having at least one connection with other infected agents:

$$S_{\tau+1} = \{m \in N | \exists\, k \in S_\tau\, g_{km} = 1\}. \tag{3.26}$$

Player A counteracts infection spreading in the following way. He/she performs the periodic scanning of the whole network and detects the set of all infected agents. Let the scanning process be instantaneous and perfect in the sense of no errors. Player A is able to stop further spread of detected infection immediately.

The strategy of player B in this game is the choice of a unique agent $j \in N$ for the initial infection of.

The strategy of player A is the choice of the scanning period, i.e., a nonnegative integer i, which has the following interpretation. For the scanning period $i = 1$ and the opponent's strategy j, only agent j becomes infected. For $i = 2$, the infected set includes agent j and also all the agents connected with him/her (i.e., agents $m \in N$ such that $g_{mj} = 1$). Let the strategy set of player A also contain ∞ ("infinite" scanning period), which indicates of no monitoring.

Under the strategies i and j of players A and B, respectively, all infected agents form the set S_i defined in i steps from the recursive Eq. (3.26) with the initial condition $S_1 = \{j\}$. Denote this set by $\delta(i, j)$.

Also make the following assumption on the decision-making of both players. Players A and B choose their strategies simultaneously and independently, i.e., a normal-form game is played.

Let us describe the payoffs of both players under their strategies (i, j).

Within the framework of the current model, suppose

(1) each agent $k \in N$ possesses some value for both players, a_k for player A and b_k for player B;
(2) the monitoring costs of player A with the scanning period i are c_i.

Then the payoffs of players A and B under their strategies (i, j) make up

$$f_{ij} = -\sum_{k \in \delta(i,j)} a_k - c_i, \tag{3.27}$$

and

$$h_{ij} = \sum_{k \in \delta(i,j)} b_k, \tag{3.28}$$

respectively.

For making the model complete, we have to accept some hypotheses about the awareness of both players. Let the structure of the social network (i.e., the communication matrix G) as well as the parameters a_k, b_k, $k \in N$, and c_i, $i = 0, 1, \ldots$, be the common knowledge [168] of players A and B.

Reduction to bimatrix game. In the previous paragraph, we have rigorously defined the strategies, awareness and payoffs of players A and B, thereby formalizing the model of informational confrontation in a social network. However, expressions (3.27) and (3.28) can be inconvenient to study particular cases. So we will design an associated bimatrix game (e.g., see [80]) in which an element (i, j) of the payoff matrix is a corresponding pair (f_{ij}, h_{ij}),

The reduction procedure to a bimatrix game employs a well-known property of the matrix G (e.g., see [97]) as follows. An element (k, j) of the matrix G^i (where $k \neq j$)[5] is nonzero if and only if the distance between the vertexes k and j does not exceed i. (Recall that the distance between two vertexes is the number of edges in a minimal path connecting them).

Consider the following sequence of matrices of dimensions $n \times n$:

$$Q_1 = E, \; Q_i = \varphi\left(G^{i-1}\right), \; i = 2, 3, \ldots, \tag{3.29}$$

where E denotes an identity matrix while an operator φ transforms all nonzero elements of a matrix into 1. Obviously, all unity elements in column j of the matrix Q_i are exactly in the rows $k \in N$ for which agent k belongs to the set $\delta(i, j)$.

Designate as f_i and h_i the rows i of the payoff matrices of players A and B, respectively. With the notations $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, and $e = (1, \ldots, 1)$ (a unit vector of dimension n), for $i = 1, 2, \ldots$ we may write

$$\begin{cases} f_i = -aQ_i - c_i e, \\ h_i = bQ_i. \end{cases} \tag{3.30}$$

Also the payoff bimatrix includes a row corresponding to the strategy $i = \infty$ (no monitoring). In this case, the monitoring costs of player A are 0.

[5]The element (k, k) of the matrix G^i is nonzero for any k and i.

Thus, the payoffs of both players can be calculated successively, row by row, using formulas (3.29) and (3.30).

By assumption, the payoff bimatrix has a finite number of rows under the condition

$$c_1 \geq c_2 \geq \ldots \tag{3.31}$$

(inequality (3.31) actually means that monitoring costs are a nonincreasing function of the scanning period). Indeed, the matrix Q_i satisfies the identity $Q_{d+1} = Q_{d+2} = \ldots$, where d is the largest diameter (i.e., the maximal distance between two vertexes) of the connected components. This leads to the relationships $f_{d+1} \leq f_{d+2} \leq \ldots$, see (3.30). Formally speaking, the monitoring strategies $i = d + 1$, $d + 2$, … are surely non-optimal for player A and dominated by the strategy $i = \infty$. The practical interpretation is as follows: if the period between two scans is sufficiently large so that player B infects the whole network (or any connected component of an unconnected network), then the monitoring procedure becomes non-beneficial to player A.

For calculating d at each step of the algorithm, it is necessary to check the condition $Q_{i+1} = Q_i$; if it holds for some index i, then $d = i + 1$. In this case, the dimensions of the payoff matrices f and h are $(d + 1) \times n$. Assume the last row of the payoff bimatrix (row $(d + 1)$) corresponds to the strategy $i = \infty$ and has the form

$$(-a\,Q_{d+1}; b\,Q_{d+1}). \tag{3.32}$$

Example 3.4 A social network consists of three agents, see Fig. 3.3 for the connections and indexes of agents.

The communication matrix of this network is $G = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Let the agents have the same value for both players, $a = b = (4; 1; 5)$. The monitoring costs of player A are given by $c_1 = 3$, $c_2 = 1$, and $c_i = 0.5$ for $i \geq 3$.

Taking into account (3.32), formulas (3.29) and (3.30) give

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ Q_2 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ Q_i = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ i \geq 3,\ d = 2,$$

$$(f, h) = \begin{pmatrix} (-7;\ 4) & (-4;\ 1) & (-8;\ 5) \\ (-6;\ 5) & (-11;\ 10) & (-7;\ 6) \\ (-10;\ 10) & (-10;\ 10) & (-10;\ 10) \end{pmatrix}. \tag{3.33}$$

Because $d = 2$, the values c_i for $i \geq 3$ make no sense.

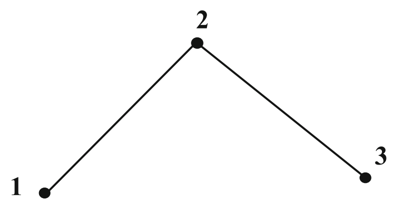

Fig. 3.3 Network in example 3.4

Bimatrix (3.33) fully describes the informational confrontation under study. Note that the informational confrontation game (3.33) has no pure strategy Nash equilibrium. •

Networks in form of complete graphs. Consider the model of informational confrontation in a social network described by a complete graph. (Recall that a graph is complete if any two vertexes are connected by an edge.)

The next example differs from Example 3.4 in the structural properties of the social network only.

Example 3.5 A social network consists of three agents, as illustrated in Fig. 3.4.

The communication matrix associated with this network has the form $G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. The other parameters are the same as in Example 3.4: $a = b = (4;\ 1;\ 5)$, $c_1 = 3$; $c_2 = 1$; $c_i = 0{,}5$, $i \geq 3$.

Using formulas (3.29), (3.30), and (3.32), we find

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ Q_i = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},\ i \geq 2,\ d = 1,$$
$$(f, h) = \begin{pmatrix} (-7;\ 4) & (-4;\ 1) & (-8;\ 5) \\ (-10\ \ 10) & (-10;\ 10) & (-10;\ 10) \end{pmatrix}. \tag{3.34}$$

Game (3.34) has precisely one Nash equilibrium in the class of pure strategies: $i = 1$, $j = 3$. In other words, player A chooses the minimal scanning period while player B chooses agent 3 for initial infection. •

As it turns out, any game of informational confrontation has an equilibrium if the communication graph of its social network is complete (sufficient condition).

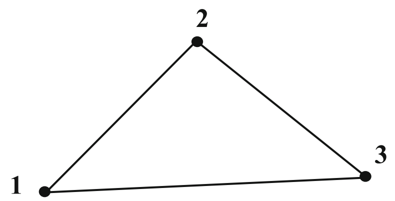

Fig. 3.4 Network in example 3.5

Proposition 3.3 There exists at least one Nash equilibrium in an arbitrary game of informational confrontation over a complete communication graph.

Proof of Proposition 3.3. Because $d = 1$ for a complete graph, then the bimatrix of this game has dimensions $2 \times n$ and the second row consists of the identical elements.

Each bimatrix of this type satisfies one of the following cases.

(1) There exists an index $j \in N$ such that $f_{2j} \geq f_{1j}$. Then the pair $(2; j)$ is a Nash equilibrium.
(2) For all $j \in N$, the inequality $f_{2j} < f_{1j}$ holds. Then a Nash equilibrium is the pair $(1; j)$, where $j \in \operatorname{Arg} \max_{k \in N} h_{1k}$.

So the game surely has at least one Nash equilibrium. •

Consequently, we have considered the informational confrontation of two players in a social network. This problem has been reduced to an associated bimatrix game. For a particular case of the social networks described by complete graphs, we have proved the existence of at least one Nash equilibrium in this game. However, will the agents gain some advantage by performing strategic reflexion in the bimatrix game (or any other associated game)? As is well-known, strategic reflexion is the process and result of agents' thinking about the actions chosen by their opponents. Interestingly, the answer to this question is affirmative!

Strategic reflexion of agents. A key issue of game theory is the models of proper actions[6] of agents in different situations. A stable set of actions in a certain sense is called a *game solution*, which emphasizes the importance of such analysis.

Since the agent's payoff (the value of his/her goal function) depends on the actions of other agents, the agent's choice is mostly determined by how he/she considers (or not) the possible thinking of the opponents in the course of their decision-making. In other words, the agent's choice is mostly determined by his/her *strategic reflexion*. For example, an agent can make decisions based on his/her own goal function only, ignoring possible actions of the opponents (the strategic reflexion of rank 0). If all agents follow this rule, the concept of *maximal guaranteed result* is a natural solution of the game: each agent maximizes his/her worst-case result under any possible actions of the opponents.

If an agent believes that the opponents have reflexion rank 0, then the rank of his/her own strategic reflexion is 1. In this case, the agent chooses the best action (which maximizes his/her goal function), expecting that the opponents will choose their guaranteeing strategies.

If an agent believes that the opponents have the strategic reflexion of rank 2, then the rank of his/her own strategic reflexion is 3, and so on. Thus, an agent of reflexion rank k considers the opponents to be of reflexion rank $(k - 1)$. Choosing any nonzero finite reflexion rank, an agent believes that the opponents perform

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[6]We consider normal-form games in which the agents choose their actions one-time, simultaneously and independently of each other. In more complicated setups (e.g., multistep games), it is necessary to discriminate between the agent's action and strategy.

other strategic reflexion. Choosing the Nash equilibrium, an agent believes that all participants of the game perform the same strategic reflexion.

Consider a game of two players with finite actions sets. Such games are called *bimatrix*, and the goal functions of both players are often defined by matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, which form the game matrix $(\mathbf{A}, \mathbf{B}) = (a_{ij}, b_{ij})$.

Denote by $I = \{1, 2, \ldots, m\}$ the action set of agent 1 and by $J = \{1, 2, \ldots, n\}$ the action set of agent 2. Introduce the following assumptions. Let the payoff matrices be such that each agent has a unique best response to any action of the opponent:

$$\forall j \in J : \left|\operatorname{Arg}\max_{i \in I} a_{ij}\right| = 1; \forall i \in I : \left|\operatorname{Arg}\max_{j \in J} b_{ij}\right| = 1. \tag{3.35}$$

Here and in the sequel, $|M|$ denotes the cardinality of a set M.

Moreover, assume the maximal guaranteed result of each agent is achieved exactly at one action:

$$\left|\operatorname{Arg}\max_{i \in I} \min_{j \in J} a_{ij}\right| = \left|\operatorname{Arg}\max_{j \in J} \min_{i \in I} b_{ij}\right| = 1. \tag{3.36}$$

Conditions (3.35) and (3.36) guarantee a univocal correspondence between the agent's reflexion rank and his/her action are presumed valid.

Each agent can choose a finite rank of his/her strategic reflexion, which results in a corresponding action. Agent 1 of reflexion rank 0 chooses the guaranteeing strategy—the action $i_0 = \arg\max_{i \in I} \min_{j \in J} a_{ij}$; of reflexion rank $k \geq 1$, the action $i_k = \arg\max_{i \in I} a_{ij_{k-1}}$

The same formulas apply to agent 2: $j_0 = \arg\max_{j \in J} \min_{i \in I} b_{ij}$ (reflexion rank 0) and $j_k = \arg\max_{j \in J} b_{i_{k-1}j}$ (reflexion rank $k \geq 1$).

The following result is the case.

Proposition 3.4 [168]. In bimatrix games, an unbounded growth of the reflexion rank is a priori irrational, i.e., there exists a reflexion rank such that higher reflexion ranks yield the same actions of all agents. The maximal rational reflexion rank does not exceed max $\{\min\{n, m + 1\}, \min\{m, n + 1\}\}$.

By Proposition 3.4, the set of admissible actions on the choice of reflexion ranks is finite. So we may pass from the original game to *the game of ranks* in which the agent's strategy is to choose the rank of his/her strategic reflexion (see Table 3.1).

Table 3.1 Reflexion ranks and actions of agents

Rank k	0	1	…	R
The action of agent 1	i_0	i_1	…	i_R
The action of agent 2	j_0	j_1	…	j_R

The upper estimate for the number of possible pairwise distinguishable strategies makes up $R = |I| \times |J| = m \times n$. Then the original bimatrix game can be transformed into a bimatrix game of dimensions $R \times R$.

Obviously, some rows and columns in the new matrix possibly coincide with each other (i.e., the choice of different ranks by agents leads to the same action in the original game). Identifying such rows and columns, we obtain the matrix of the new game, further called *the choice game for the rank of strategic reflexion* (or simply *the game of ranks*).

Because $i_k \in I$ and $j_k \in J$, all actions of the agents in the game of ranks correspond to their actions in the original game. Hence, we have the following fact.

Proposition 3.5 In the game of ranks, the payoff matrix is a submatrix of the matrix of the original bimatrix game.

Proposition 3.2 suggests that the transition to the game of ranks possibly eliminates some equilibria (i.e., they will disappear in the new matrix).

Example 3.6 Consider a bimatrix game given by

$$\begin{pmatrix} (2,3) & (0,0) & (3,2) \\ (0,0) & (4,4) & (0,1) \\ (3,2) & (1,0) & (2,3) \end{pmatrix}.$$

To construct the matrix of the game of ranks, analyze the choice of agents under different reflexion ranks (see Table 3.2).

Therefore, this matrix takes the form

$$\begin{pmatrix} (2,3) & (3,2) \\ (3,2) & (2,3) \end{pmatrix}.$$

Clearly, the equilibrium pair of payoffs (4, 4) has been lost after the transition to the game of ranks. •

Hence, the question is: Will transition to the game of ranks yield new equilibria? Actually, no.

Proposition 3.6a For an arbitrary bimatrix game, the transition to the game of ranks yields no new equilibria.

Proof of Proposition 3.6a. As before, let I be the action set of agent 1 and J the action set of agent 2. Denote by $I' \subseteq I$ and $J' \subseteq J$ the action sets of these agents in the game of ranks.

Consider a pair of actions (i_u, j_v), $i_u \in I'$, $j_v \in J'$, which is the equilibrium in the game of ranks.

Table 3.2 Reflexion ranks and actions of agents in example 3.6

Rank k	0	1	2	...
The action of agent 1	3	1	3	…
The action of agent 2	3	1	3	…

First, demonstrate that j_v is the best response of agent 2 to the action i_u of agent 1 in the original game. Indeed, the best response over the set J also belongs to J' by the design procedure of the game of ranks. Thus, the best response over the set J' coincides with that over the set J. By the definition of equilibrium[7], the best response on the set J' is j_v.

By analogy, i_u is the best response of player 1 to the strategy j_v of player 2 in the original game. Therefore, the pair of actions (i_u, j_v) forms an equilibrium in the original game.

Because the choice of the equilibrium pair is assumed to be arbitrary, any equilibrium in the game of ranks becomes an equilibrium in the original game (i.e., new equilibria do not appear). •

So the transition to the game of ranks does not produce new equilibria (Proposition 3.6a). Moreover, the existing equilibria may even disappear (Example 3.6). The number of equilibria in the game of ranks satisfies the following estimate that relies on (3.35) and (3.36).

Proposition 3.6b Under conditions (3.35) and (3.36), the game of ranks has at most two equilibria.

Proof of Proposition 3.6b Assume the game of ranks has three different equilibria: (i_u, j_v), $(i_{u'}, j_{v'})$, and $(i_{u''}, j_{v''})$. By Proposition 3.3, they are equilibria in the original game. Then, on the strength of (3.35), $i_u \neq i_{u'} \neq i_{u''}$. Without loss of generality, let $u = \max\,[u; u'; u'']$. In an equilibrium, the action of an agent gives the best response to the opponent's action and hence $i_u = i_{v+1} = i_{u+2} = i_{v+3}$- $= i_{u+4} = \ldots$; $j_v = j_{u+1} = j_{v+2} = \ldots$. Similar equalities hold for $i_{u'}$ and $i_{u''}$. Consequently, $i_{u+1} = i_{u'}$ and $i_u = i_{u''}$. On the other hand, this implies $i_u = i_{u''}$, which contradicts the conditions above. The proof of Proposition 3.6b is complete. •

Interestingly, sometimes any outcome of the agents' game leads to a better result than an equilibrium, as illustrated below.

Example 3.7 Consider the bimatrix game defined by

$$\begin{pmatrix} (6,10) & (0,0) & (10,6) \\ (0,0) & (5,5) & (0,1) \\ (10,6) & (1,0) & (6,10) \end{pmatrix}.$$

Here the equilibrium leads to the pair of payoffs (5, 5). Obviously, this is worse than any outcome in the game of ranks (for both agents):

$$\begin{pmatrix} (6,10) & (10,6) \\ (10,6) & (6,10) \end{pmatrix}. \quad \bullet$$

Future investigations can be focused on the games of ranks "superstructed" over bimatrix games without pure strategy Nash equilibria.

[7]Recall that the matter concerns Nash equilibria.

3.3 Informational Confrontation in Mob Control

This section considers the problematique of informational confrontation, mostly on the example of mobs as active network structures. First, we describe a mob model using the results of the paper [27] and also informational confrontation within the stochastic models of mob control on the basis of [29]. Next, we present original analysis results for the game-theoretic models of informational confrontation in terms of normal-form games (including a characterization of Nash equilibria and equilibria in secure strategies), hierarchical games and reflexive games, see [163]. Numerous examples illustrate how these equilibria depend on the model parameters in analytic form.

Mob model

Denote by $N = \{1, \ldots, n\}$ a finite set of agents. Agent $i \in N$ in a mob is characterized by his/her *decision* $x_i \in \{0; 1\}$ (being active or passive) and *threshold* $\theta_i \in [0, 1]$ that defines the agent's choice under a given *opponents' action profile* (the vector x_{-i} comprising the decisions of the other agents). In other words, agent i chooses his/her action as the *best response* (BR) to the existing opponents' action profile:

$$x_i = BR_i(x_{-i}) = \begin{cases} 1 \text{ if } \frac{1}{n-1}\sum\limits_{j \neq i} x_j \geq \theta_i, \\ 0 \text{ if } \frac{1}{n-1}\sum\limits_{j \neq i} x_j < \theta_i. \end{cases} \tag{3.37}$$

The behavior described by (3.37) is called *threshold behavior*, see the pioneering paper [87] and also the surveys in [34, 147].

Consider the following *dynamic model of collective behavior* [27]. At the initial (zero) step, all agents are passive. At each subsequent step, the agents act simultaneously and independently according to the procedure (3.37). Introduce the notation $Q_0 = \varnothing$,

$$Q_1 = \{i \in N | \theta_i = 0\}, Q_k = Q_{k-1} \cup \{i \in N | \# Q_{k-1} \geq n\,\theta_i\},\ k = 2, \ldots, n-1, \tag{3.38}$$

where # means set cardinality, Q_k is the set of active agents at step k. Obviously, $Q_0 \subseteq Q_1 \subseteq \ldots \subseteq Q_n \subseteq N$. For the agents' threshold vector $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$, calculate the index $q(\theta) = \min\{k = \overline{0, n-1} | Q_{k+1} = Q_k\}$. Define a *collective behavior equilibrium* (CBE) [27] by

$$x_i^*(\theta) = \begin{cases} 1 \text{ if } i \in Q_{q(\theta)}, \\ 0 \text{ if } i \in N \backslash Q_{q(\theta)}, \end{cases} \quad i \in N.$$

The value $x^* = \frac{\#Q_{q(\theta)}}{n} = \frac{1}{n}\sum_{i \in N} x_i^*(\theta) \in [0, 1]$ reflects the "mob state," i.e., the share of active agents in the CBE. As established in [24, 27], the CBE is a Nash equilibrium in the agents' game with the best response (3.37).

Suppose the number of agents is large. Denote by $F(\cdot)$: [0, 1] → [0, 1] *the distribution function of the agents' thresholds*, a nondecreasing function defined on the unit segment (the set of all admissible thresholds) that is left-continuous and possesses the right-hand limit at each point of its definitional domain.

Assume we know the share x^k of active agents at step k, where $k = 0, 1, \ldots$ Further behavior of the agents satisfies the following recurrent expression [24, 87]:

$$x^{l+1} = F\left(x^l\right), \tag{3.39}$$

where $l = k, k + 1, \ldots$ are subsequent steps.

The equilibria of the discrete dynamic system (3.39) are determined by the initial point x^0 (on default, $x^0 = 0$) and also by the intersection points of the curve $F(\cdot)$ and the bisecting line of quadrant I:

$$F(x) = x. \tag{3.40}$$

Interestingly, the trivial equilibrium is 1 by the properties of a distribution function.

The potentially stable equilibria of system (3.39) are the points at which the curve $F(\cdot)$ crosses the bisecting line by approaching it "from left and top." Designate as $y = \inf\{x : x \in (0, 1], F(x) = x\}$ the least nonzero root of Eq. (3.40).

In accordance with (3.38) and (3.39), the collective behavior equilibrium [93] is the point

$$x^* = \begin{cases} y \text{ if } \forall z \in [0, y] F(z) \geq z, \\ 0 \text{ otherwise.} \end{cases} \tag{3.41}$$

Model of informational confrontation

Consider a mob as an object controlled by two subjects (*Principals*). The behavior of the dynamic system (3.39) that describes the evolution of the share of active agents is determined by the distribution function $F(\cdot)$ of the agents' thresholds. So we will analyze the control actions that change this distribution function.

Note that the paper [27] defined the set (share) of the initially excited agents or/and the distribution function of their thresholds that implement a required equilibrium. Within the models studied below, the agents are excited "independently," as illustrated by formula (3.38).

The following models of control applied by the Principals to the distribution function of the agents' thresholds were proposed in [29].

Model I. Suppose a control action nullifies the threshold of each agent independently from the other agents with a same probability $\alpha \in [0, 1]$ for all the agents. In accordance with (3.37), the agents having zero thresholds choose unit actions regardless of the actions of the other agents. Hence, the parameter α is interpreted as the share of the initially *excited* agents (provokers) [29].

Now, assume a control action makes the threshold of each agent equal to 1 independently from the other agents with a same probability $\beta \in [0, 1]$ for all agents. In accordance with (3.37), the agents having unit thresholds are passive, and the parameter β gives the share of the initially "*immunized*" agents [29].

The paper [29] also examined the case of informational confrontation in the following setup. There are *Principal 1* "exciting" a share $\alpha \in [0, 1]$ of agents and *Principal 2* "immunizing" a share $\beta \in [0, 1]$ of agents (alternatively, each agent can be independently excited or/and immunized with a corresponding probability by the other Principal). For the sake of definiteness, the threshold of any agent that is excited and immunized simultaneously remains the same. Other setups are also possible, which would yield different results. Under the assumption about "infinitely" many agents, it was demonstrated in [29] that the distribution function of the agents' thresholds has the form

$$F_{\alpha,\beta}(x) = \begin{cases} \alpha(1-\beta) + (1-\alpha-\beta+2\alpha\beta)\,F(x), \; x \in [0,1), \\ 1, \; x = 1. \end{cases} \tag{3.42}$$

Denote by $x^*(\alpha, \beta)$ the CBE (3.41) corresponding to the distribution function (3.42) and by $y_{\alpha,\beta} = \inf\{x : x \in (0,1],\; F_{\alpha,\beta}(x) = x\}$ the least nonzero root of the equation $F_{\alpha,\beta}(x) = x$. Then

$$x^*(\alpha,\beta) = \begin{cases} y_{\alpha,\beta} & \text{if } \forall z \in [0, y_{\alpha,\beta}] : F_{\alpha,\beta}(z) \geq z, \\ 0 & \text{otherwise.} \end{cases} \tag{3.43}$$

Using expressions (3.40) and (3.42), we find the control pairs (α, β) implementing the given CBE (3.43).

Designate as

$$\Omega(x) = \left\{(\alpha,\beta) \in [0;1]^2 \middle| x^*(\alpha,\beta) = x\right\}$$

the set of the control pairs *implementing* a given value $x \in [0, 1]$ as the CBE.

Let $W = \bigcup_{(\alpha,\beta)\in[0;1]^2} x^*(\alpha,\beta)$ be *the set of attainable equilibria*. The game-theoretic analysis below relies on another important result of [29] as follows: the CBE $x^*(\alpha, \beta)$ is monotonically (nonstrictly) increasing in α and monotonically (nonstrictly) decreasing in β; for strict monotonicity, a sufficient condition is defined by

$$F(0) > 0,\; F(1-0) < 1. \tag{3.44}$$

The paper [29] also obtained sufficient conditions, in terms of the properties of the distribution function, under which a given point $x \in [0, 1]$ is implemented as the CBE using certain control actions $(\alpha, \beta) \in [0, 1]^2$.

Model II. Now, consider informational confrontation: Principal 1 adds k provokers with zero thresholds while Principal 2 adds l immunizers with unit thresholds to the set N. For a sufficiently large n, we will employ the continuous approximation $\delta = k / n$, $\gamma = l / n$, assuming that $\delta, \gamma \in \mathbb{R}^1_+$.

In this case [29], the distribution function of the agents' thresholds has the form

$$F_{\delta,\gamma}(x) = \begin{cases} \dfrac{\delta + F(x)}{1 + \delta + \gamma}, \ x \in [0, 1), \\ 1, \ x = 1. \end{cases} \tag{3.45}$$

Denote by $x^*(\delta, \gamma)$ the CBE (3.41) corresponding to the distribution function (3.45). Let $y_{\delta,\gamma} = \inf\{x : x \in (0, 1], \ F_{\delta,\gamma}(x) = x\}$ be the least nonzero root of the equation $F_{\delta,\gamma}(x) = x$. Then

$$x^*(\delta, \gamma) = \begin{cases} y_{\delta,\gamma} & \text{if } \forall z \in [0, y_{\delta,\gamma}] : F_{\delta,\gamma}(z) \geq z, \\ 0 & \text{otherwise.} \end{cases} \tag{3.46}$$

For Model II, it was established in [29] that $W = (0, 1]$ and, moreover, $W = [0, 1]$ if $F(0) = 0$. Designate as

$$\Lambda(x) = \{(\delta, \gamma) \in \mathbb{R}^2_+ \,|x^*(\delta, \gamma) = x\}$$

the set of all control pairs implementing a given value $x \in [0, 1]$ as the CBE.

To explore the game-theoretic models of interaction between the Principals, we will need a result that is proved similarly to Assertions 3 and 4 in the paper [29].

Proposition 3.7 *For Model II, the CBE* $x^*(\delta, \gamma)$ *has the following properties*:

(1) *monotonic (nonstrict) increase in* δ*; for strict monotonicity, a sufficient condition is* $F(1 - 0) < 1$ *or* $\gamma > 0$;
(2) *monotonic (nonstrict) decrease in* γ*; for strict monotonicity, a sufficient condition is* $F(0) > 0$ *or* $\delta > 0$.

Example 3.8 Consider the uniform distribution of the agents' thresholds, i.e., $F(x) = x$. Here $x^*(\delta, \gamma) = \delta / (\delta + \gamma)$ and $\Lambda(x) = \{(\delta, \gamma) \in \mathbb{R}^2_+ \,|\gamma/\delta = (1/x - 1)\}$. •

As a digression, note an important feature of the socioeconomic and organizational systems with several subjects who are interested in certain states of a controlled system (e.g., a network of interacting agents) and applying control actions to it (the systems with distributed control [78, 165, 169]). In such systems, like in the current setup, there exists an interaction between the subjects, which is termed *informational confrontation* if they have informational influence on the controlled object (see the surveys in the current section and in [160] as well as Sect. 3.1).

These situations are often described by a normal-form game of the Principals; the strategies chosen by the latter define the parameters of the agents' game [165].

For example, consider the models of informational confrontation in social networks (see the preceding sections of this chapter and also [93]). As mentioned in [158], more complicated situations are also possible in which the control actions have no "symmetry." In the "attack/defense" situation, Principal 1 influences the initial states of the agents whereas Principal 2 modifies the structure of the relations among them or/and their thresholds (the latter acts simultaneously with the opponent or right after him/her, being aware of his/her choice). Such situations can be characterized in terms of hierarchical games.

In what follows, we will consider a series of game-theoretic models of interaction between the Principals whose informational influences on a mob are defined by expressions (3.42) and (3.43) (Model I) or expressions (3.45) and (3.46) (Model II).

Normal-form game of Principals

Model I. Two Principals apply an informational influence on a mob by playing a *normal-form game*. That is, Principal 1 and Principal 2 choose their strategies $\alpha \in [0, 1]$ and $\beta \in [0, 1]$, respectively, one-time, simultaneously and independently of each other. The *goal functions* of Principals 1 and 2 have the form

$$f_\alpha(\alpha, \beta) = H_\alpha(x^*(\alpha, \beta)) - c_\alpha(\alpha), \tag{3.47}$$

$$f_\beta(\alpha, \beta) = H_\beta(x^*(\alpha, \beta)) - c_\beta(\beta). \tag{3.48}$$

Moreover, the *payoff* $H_\alpha(\cdot)$ of Principal 1 is an increasing function as he/she seeks for maximizing the number of the excited agents; the payoff $H_\beta(\cdot)$ of Principal 2 is a decreasing function because his/her interests are quite the opposite. Both cost functions, $c_\alpha(\cdot)$ and $c_\beta(\cdot)$, are strictly increasing and $c_\alpha(0) = c_\beta(0) = 0$.

The described game belongs to the class of normal-form ones, and several typical questions of game theory [80, 153] arise immediately. What is the *Nash equilibrium* (α^*, β^*) in the agents' game? For which strategy profiles is the Nash equilibrium dominating the *status quo profile*, i.e., the CBE without control (i.e., when do the conditions $f_\alpha(\alpha^*, \beta^*) \geq f_\alpha(0, 0)$ and $f_\beta(\alpha^*, \beta^*) \geq f_\beta(0, 0)$) hold? What is the structure of the set of Pareto efficient strategy profiles? When does *a dominant strategy equilibrium* (DSE) exist? And so on.

Denote by $f(\alpha, \beta) = f_\alpha(\alpha, \beta) + f_\beta(\alpha, \beta)$ *the utilitarian collective utility function* (CUF) [151]. The pair of strategies $(\hat{\alpha}; \hat{\beta}) = \arg\max_{(\alpha,\beta)\in[0;1]^2} f(\alpha, \beta)$ will be called *the utilitarian solution*.

The results obtained in the paper [29] and Proposition 3.7 are crucial for game-theoretic analysis, as explained below. The goal functions (3.47) and (3.48) of the Principals depend on their strategies (α and β or δ and γ) and on the CBE, which is in turn dependent on these strategies. The monotonic dependence of the CBE on the strategies of the Principals (if necessary, its continuity can be checked in a specific case), as well as the implementability of the whole unit segment as the CBE via the appropriately chosen strategies, allow "translating" the properties of the goal and cost functions on the dependence of these parameters directly from the strategies of the Principals. For example, if $H_\delta(x^*(\delta, \gamma))$ is an increasing function of x^*, then by Proposition 3.7 the payoff of Principal 1 is an increasing function of his/her strategy, and so on.

An elementary case is *the antagonistic game*: Principal 1 seeks for maximizing the number of excited agents while Principal 2 has the opposite interests. For $c_\alpha(\cdot) \equiv 0$ and $c_\beta(\cdot) \equiv 0$ (no control costs), expressions (3.47) and (3.48) yield

$$\hat{f}_\alpha(\alpha, \beta) = x^*(\alpha, \beta), \hat{f}_\beta(\alpha, \beta) = 1 - x^*(\alpha, \beta). \tag{3.49}$$

Clearly, $f(\alpha, \beta) \equiv 1$. As $x^*(\alpha, \beta)$ does not decrease in α and increase in β, we naturally arrive at Proposition 3.8. Like its "analogs" for Model II (see Propositions 3.10 and 3.11 below), this proposition seems trivial in some sense, following directly from the monotonicity of the goal functions of agents in their actions. On the other hand, Proposition 3.8 proves the existence of the DSE and gives a method to calculate it in the degenerate cases.

Proposition 3.8 *For Model I described by the antagonistic game with zero control costs, there exists the DSE* $\alpha^{DSE} = 1$, $\beta^{DSE} = 1$.

Note that, in this equilibrium, the distribution function of the agents' thresholds coincides with the initial distribution function, i.e., $F_{1,1}(x) \equiv F(x)$. Hence, the CBE remains invariable, "matching" the status quo profile.

Example 3.9 Choose $F(x) = x$; then

$$x^{I^*}(\alpha, \beta) = \frac{\alpha(1-\beta)}{\alpha + \beta - 2\alpha\beta}. \tag{3.50}$$

Calculate the partial derivatives

$$\frac{\partial x^{I^*}(\alpha, \beta)}{\partial \alpha} = \frac{\beta(1-\beta)}{(\alpha + \beta - 2\alpha\beta)^2}, \quad \frac{\partial x^{I^*}(\alpha, \beta)}{\partial \beta} = -\frac{\alpha(1-\alpha)}{(\alpha + \beta - 2\alpha\beta)^2},$$

which shows that $x^{I*}(\alpha, \beta)$ is increasing in the first argument and decreasing in the second argument for any admissible values of the other argument. Therefore, under zero control costs, the DSE in the Principals' game with the goal functions (3.49) is the unit strategies $\alpha^{DSE} = 1$, $\beta^{DSE} = 1$. Naturally, this point also represents the Nash equilibrium (NE) in the Principals' game. In the current example, we have $W = [0, 1]$. The DSE implements the same mob state as in the absence of control. •

Now, consider the case of nonzero control costs.

Proposition 3.9 *For Model I with* $W = [0, 1]$ *and condition* (3.44), *let* $x^*(\alpha, \beta)$ *be a continuous function, the payoff functions of the Principals be bounded, linear or concave in their strategies and the cost functions be convex. Then there exists a Nash equilibrium in the Principals' game.*

The trivial Proposition 3.9 (and its "analog" for Model II—Proposition 3.12) directly follows from the sufficient conditions of Nash equilibrium existence in the continuous games, see [80, 153].

The example below has a unique Nash equilibrium.

Example 3.10 Choose $F(x) = x$, $H_\alpha(x) = x$, $H_\beta(x) = 1 - x$, $c_\alpha(\alpha) = -\ln(1 - \alpha)$, and $c_\beta(\beta) = -\lambda \ln(1 - \beta)$. The first-order optimality conditions yield $\beta = (1/\lambda)\ \alpha$. For $\lambda = 1$, we obtain $\alpha^* = 1/4$, $\beta^* = 1/4$. In this case,

$$x^{I^*}(\alpha^*, \beta^*) = 1/2,\ f_\alpha(\alpha^*, \beta^*) = f_\beta(\alpha^*, \beta^*) \approx -0.2.$$

Interestingly, in the equilibrium both Principals have smaller values of their goal functions than in *the status quo profile* (0; 0) because $f_\alpha(0, 0) = 1$ and $f_\beta(0, 0) = 0$. Here the utilitarian solution is also the zero strategy profile. •

Model II. Consider the goal functions of Principals 1 and 2 of form (3.47) and (3.48), respectively, except that α is replaced by δ and β by γ.

Proposition 3.10 *For Model II described by the antagonistic game with zero control costs, there exist no finite DSE or NE in the Principals' game.*

This proposition is immediate from the boundedness of the admissible strategy sets of the Principals and from the monotonicity of $x^*(\delta, \gamma)$ in both variables (see Proposition 3.7). In addition, these properties guarantee the following result.

Proposition 3.11 *For Model II described by the antagonistic game with zero control costs, let the admissible strategy sets of the Principals be bounded*: $\delta \leq _{\max}$, $\gamma \leq \gamma_{\max}$. *Then there exists the DSE* $\delta^{DSE} = \delta_{\max}$, $\gamma^{DSE} = \gamma_{\max}$ *in the Principals' game.*

Consider the case of nonzero control costs.

Proposition 3.12 *For Model II under the hypotheses of Proposition 3.7, let* $x^*(\delta, \gamma)$ *be a continuous function, the payoff functions of the Principals be bounded, linear or concave in their strategies and also the cost functions be convex with the zero derivatives at the zero point and infinite growth as the argument tends to infinity. Then there exists a finite Nash equilibrium in the Principals' game.*

The proof of Proposition 3.12 is straightforward. Under the above hypotheses, the goal functions of the Principals are concave in their strategies and take non-negative values on the bounded value set of the arguments. So a Nash equilibrium exists in this continuous game by the sufficient conditions [80].

Example 3.11 Choose $F(x) = x$, $H_\delta(x) = x$, $H_\gamma(x) = 1 - x$, $c_\delta(\delta) = \delta^2$, and $c_\gamma(\gamma) = \lambda^2\ \gamma^2$. In accordance with Example 3.8, the CBE is $x^*(\delta, \gamma) = \delta\ /\ (\delta + \gamma)$. The goal functions of the Principals have the form

$$f_\delta(\delta, \gamma) = \delta/(\delta + \gamma) - \delta^2, \tag{3.51}$$

$$f_\gamma(\delta,\ \gamma) = 1 - \delta/(\delta + \gamma) - \lambda^2\gamma^2. \tag{3.52}$$

The goal functions (3.51) and (3.52) are concave in δ and γ, respectively. The first-order optimality conditions yield the Nash equilibrium

$$\delta^* = \sqrt{\frac{\lambda}{2}}\frac{1}{1+\lambda}, \; \gamma^* = \frac{1}{\sqrt{2\lambda}}\frac{1}{1+\lambda}.$$

In this case, the CBE is $x^*(\delta^*, \gamma^*) = \frac{\lambda}{1+\lambda}$, and in the NE the goal functions take the values $f_\delta(\delta^*, \gamma^*) = \frac{\lambda(1+2\lambda)}{2(1+\lambda)^2}$, $f_\gamma(\delta^*, \gamma^*) = \frac{\lambda+2}{2(1+\lambda)^2}$.

The utilitarian CUF $f(\delta, \gamma) = f_\delta(\delta, \gamma) + f_\gamma(\delta, \gamma)$ achieves maximum (actually, 1) in the zero strategy profile. The value of this function in the NE is $f(\delta^*, \gamma^*) = 1 - \frac{\lambda}{(1+\lambda)^2}$. So, the term $\frac{\lambda}{(1+\lambda)^2}$ characterizes how "worse" the NE value of the utilitarian CUF is in comparison with its optimal value. •

Threshold goal functions of Principals

For practical interpretations, an important case concerns the *threshold payoff functions* of the Principals, i.e.,

$$H_{\alpha(\beta)}(x) = \begin{cases} H^+_{\alpha(\beta)} & \text{if } x \geq (\leq)\, \theta_\alpha(\theta_\beta), \\ H^-_{\alpha(\beta)} & \text{otherwise}, \end{cases} \tag{3.53}$$

where $H^+_{\alpha(\beta)} > H^-_{\alpha(\beta)}$. That is, Principal 1 obtains a higher payoff if the share of active agents is not smaller than a threshold $\theta_\alpha \in [0, 1]$; Principal 2 obtains a higher payoff if the share of active agents does not exceed a threshold $\theta_\beta \in [0, 1]$. Denote by $\hat{x}$ the CBE in the absence of the Principals' control actions: $\hat{x} = x^*(0, 0)$. We will need a pair of hypotheses as follows.

Assumption A.1 The attainability set W is the unit segment, $x^*(\alpha, \beta)$ is a strictly monotonic continuous function of its arguments, and the cost functions of the Principals are strictly monotonic.

See the corresponding sufficient conditions above or check these conditions in each specific case.

Assumption A.2 Under the zero strategy of Principal 2, Principal 1 can independently implement any CBE from $[\hat{x}, 1]$; under the zero strategy of Principal 1, Principal 2 can independently implement any CBE from $[0, \hat{x}]$. ♦

The structure of the goal functions of the Principals, together with Assumptions A.1 and A.2, directly imply the following. For Principal 1 (Principal 2), it is nonbeneficial to implement any CBE exceeding the threshold θ_α (any CBE strictly smaller than the threshold θ_β, respectively).

Model I. Define the Nash equilibrium $(\alpha^*; \beta^*)$:

$$\begin{cases} \forall \alpha \in [0, 1] : H_\alpha(x^*(\alpha^*, \beta^*)) - c_\alpha(\alpha^*) \geq H_\alpha(x^*(\alpha, \beta^*)) - c_\alpha(\alpha), \\ \forall \beta \in [0, 1] : H_\beta(x^*(\alpha^*, \beta^*)) - c_\beta(\beta^*) \geq H_\beta(x^*(\alpha^*, \beta)) - c_\beta(\beta). \end{cases}$$

First, consider the special case $\theta_\beta = \theta_\alpha = \theta$.

Introduce the notations $\alpha(\theta) = \min\{\alpha \in [0,1] | x^*(\alpha, 0) = \theta\}$ and $\beta(\theta) = \min\{\beta \in [0,1] | x^*(0, \beta) = \theta\}$.

Define the set

$$\Omega_{\alpha,\beta}(\theta) = \Big\{(\alpha;\ \beta) \in [0,1]^2 \big| x^*(\alpha, \beta) = \theta, \tag{3.54}$$

$$c_\alpha(\alpha) \le H_\alpha^+ - H_\alpha^-,\ c_\beta(\beta) \le H_\beta^+ - H_\beta^- \Big\},$$

which includes the pairs of strategies implementing the CBE θ with the following property: each Principal gains not less than by using the strategy that modifies his/her payoff (3.53). By analogy with [165, 169], set (3.54) will be called *the domain of compromise*.

By definition, if the domain of compromise is nonempty, then by implementing the CBE θ with the utilitarian CUF the agents guarantee a payoff that is not smaller than in the status quo profile $\hat{x}$. Moreover, the Principals obviously benefit nothing by implementing any other CBE (perhaps, except $\hat{x}$ or θ).

Proposition 3.13 *If $\theta_\beta = \theta_\alpha = \theta$ and Assumption A.1 holds, then there may exist NE of the two types only as follows:*

(1) (0; 0) *is the NE under the conditions*

$$\hat{x} \le \theta \text{ and } c_\alpha(\alpha(\theta)) \ge H_\alpha^+ - H_\alpha^- \tag{3.55}$$

or

$$\hat{x} \ge \theta \text{ and } c_\beta(\beta(\theta)) \ge H_\beta^+ - H_\beta^-; \tag{3.56}$$

(2) *the set of NE includes the nonempty set $\Omega_{\alpha,\beta}(\theta)$ if any.*
Furthermore, if Assumption A.2 holds, then
$(\alpha(\theta); 0)$ *is the NE under the conditions*

$$\hat{x} \le \theta \text{ and } c_\alpha(\alpha(\theta)) \le H_\alpha^+ - H_\alpha^-; \tag{3.57}$$

$(0;\ \beta(\theta))$ *is the NE under the conditions*

$$\hat{x} \ge \theta \text{ and } c_\beta(\beta(\theta)) \le H_\beta^+ - H_\beta^-. \tag{3.58}$$

Now, we will explore a relationship between the domain of compromise and the utilitarian solution. Denote by

$$C(\theta) = \min_{(\alpha,\beta)\in\Omega_{\alpha,\beta}(\theta)} \left[c_\alpha(\alpha) + c_\beta(\beta)\right] \tag{3.59}$$

the minimum total cost of the Principals to implement the CBE θ. In the case under consideration, the utilitarian solution satisfies the following conditions:

– if $\hat{x} \le \theta$, then $f(\hat{\alpha};\ \hat{\beta}) = \max\left\{H_\alpha^- + H_\beta^+;\ H_\alpha^+ + H_\beta^+ - C(\theta)\right\}$;
– if $\hat{x} \ge \theta$, then $f(\hat{\alpha};\ \hat{\beta}) = \max\left\{H_\alpha^+ + H_\beta^-;\ H_\alpha^+ + H_\beta^+ - C(\theta)\right\}$.

So, if for $\hat{x} \le \theta$ we have $C(\theta) \le H_\alpha^+ - H_\alpha^-$ while for $\hat{x} \ge \theta$ $C(\theta) \le H_\beta^+ - H_\beta^-$, then the domain of compromise includes the utilitarian solution.

The example below demonstrates a crucial role of Assumption A.2 for the NE structure.

Example 3.12 Choose $F(x) = x$, $\theta = 1/2$, $H_\alpha^- = H_\beta^- = 0$, $H_\alpha^+ = H_\beta^+ = 1$, $c_\alpha(\alpha) = -\ln(1-\alpha)$, and $c_\beta(\beta) = -\ln(1-\beta)$. Clearly, see Example 3.10, the zero strategy profile is not an NE. In accordance with the results of Example 3.9 and expressions (3.54)–(3.59):

$$\Omega_{\alpha,\beta}(1/2) = \left\{(\alpha,\beta) \in [0;1]^2 \middle| \frac{\alpha(1-\beta)}{\alpha+\beta-2\alpha\beta} = 1/2,\ \ln(1-\alpha) \ge -1,\ \ln(1-\beta) \ge -1\right\},$$

i.e., $\Omega_{\alpha,\beta}(1/2) = \left\{(\alpha,\beta) \in [0;1]^2 \middle| \alpha = \beta,\ 0 < \alpha,\ \beta \le 1 - 1/e\right\}$. In this example, the ε-optimal utilitarian solution is the Principals' strategy profile (ε, ε), where $\varepsilon \in (0, 1 - 1/e]$. •

Next, consider the general case in which the Principals' thresholds appearing in the payoff functions (3.53) are different. In terms of applications (informational confrontation), the most important relationship between the thresholds is described by

$$\theta_\beta < \hat{x} < \theta_\alpha. \tag{3.60}$$

Define the following functions:

$$C_\alpha(x,\beta) = \min_{\{\alpha\in[0,1] | x^*(\alpha,\beta)=x\}} c_\alpha(\alpha),\ C_\beta(x,\alpha) = \min_{\{\beta\in[0,1] | x^*(\alpha,\beta)=x\}} c_\beta(\beta).$$

(Whenever minimization runs on the empty set, a corresponding function will be supposed + ∞.)

Since the cost functions are nondecreasing and the payoff functions have form (3.53), the Principals do not benefit by implementing the CBE from the interval $(\theta_\beta;\ \theta_\alpha)$ in comparison to the status quo profile $\hat{x}$. Introduce another hypothesis that actually relaxes Assumption A.2.

Assumption A.3 Under the zero strategy of Principal 2, Principal 1 can independently implement the CBE θ_α; under the zero strategy of Principal 1, Principal 2 can independently implement the CBE θ_β. ♦

The result below is immediate from the definition of Nash equilibrium and the properties of the Principals' goal functions.

Proposition 3.14 *Under Assumptions* A.1, A.3 *and condition* (3.60)*, the Nash equilibria in the Principals' game have the following characterization*:

– (0; 0) *is the NE if*

$$\begin{cases} H_\alpha^+ - c_\alpha(\alpha(\theta_\alpha)) \le H_\alpha^-, \\ H_\beta^+ - c_\beta(\beta(\theta_\beta)) \le H_\beta^-; \end{cases} \tag{3.61}$$

– $(\alpha(\theta_\alpha); 0)$ *is the NE if*

$$\begin{cases} H_\alpha^+ - c_\alpha(\alpha(\theta_\alpha)) \ge H_\alpha^-, \\ H_\beta^- \ge H_\beta^+ - C_\beta(\theta_\beta,\ \alpha(\theta_\alpha)); \end{cases} \tag{3.62}$$

– $(0; \beta(\theta_\beta))$ *is the NE if*

$$\begin{cases} H_\beta^+ - c_\beta(\beta(\theta_\beta)) \ge H_\beta^-, \\ H_\alpha^- \ge H_\alpha^+ - C_\alpha(\theta_\alpha,\ \beta(\theta_\beta)). \end{cases} \tag{3.63}$$

Model II with the threshold payoff functions of the Principals is designed by analogy to Model I: it suffices to replace α by δ, and β by γ. The next examples are illustrating Proposition 3.14 for Model II.

Example 3.13 Choose $F(x) = 1/3 + 2x^2/3$, $\theta_\gamma = 0.4$, $\theta_\delta = 0.6$, $H_\delta^- = H_\gamma^- = 0$, $H_\delta^+ = H_\gamma^+ = 1$, $c_\delta(\delta) = \delta^2$, and $c_\gamma(\gamma) = \lambda^2\,\gamma^2$.

Here we easily calculate $\hat{x} = 1/2$, $\gamma(\theta_\gamma) \approx 0.1$, and $\delta(\theta_\delta) \approx 0.07$.

For $\lambda = 2$, conditions (3.61)–(3.63) all fail and hence the NE does not exist.

For $\lambda = 20$, conditions (3.61) and (3.63) fail but condition (3.62) holds. Therefore, (0.07; 0) is the NE. •

Example 3.14 For the data of Example 3.13, choose $\theta_\gamma = \theta_\delta = \theta = 0.4$ and $\lambda = 20$. In this case,

$$\Omega_{\delta,\gamma}(0,4) = \{\delta \in [0,1],\ \gamma \in [0,0.05] | \gamma = 0.1 + 1.5\,\delta\} = \emptyset.$$

Condition (3.56) is true, i.e., the trivial profile (0; 0) gives the NE. •

If there exist no Nash equilibria, an alternative approach is to find and analyze the equilibria in secure strategies (ESSs). This concept was originally suggested in the paper [109] as the equilibria in safe strategies and then restated in a simpler form (see [110, 111] for details). The ESS proceeds from the notion of a threat. There is a *threat* to a player if another player can increase his/her payoff and simultaneously decrease the payoff of the given player via a unilateral deviation. An *equilibrium in secure strategies* is defined as a strategy profile with the following properties:

– all the players have no threats;
– none of the players can increase his/her payoff by a unilateral deviation without creating a threat to lose more than he/she gains.

Under Assumptions A.1 and A.2, define the following functions:

$$C_\delta(x,\,\gamma) = \min_{\{\delta \ge 0 | x^*(\delta,\gamma)=x\}} c_\delta(\delta),\;\; C_\gamma(x,\delta) = \min_{\{\gamma \ge 0 | x^*(\delta,\gamma)=x\}} c_\gamma(\gamma).$$

Again, if minimization runs on the empty set, a corresponding function will be supposed + ∞.

Using the definition of ESS (see above and also the papers [109, 110]) together with the properties of the Principals' goal functions, we establish the following result.

Proposition 3.15 *Let Assumptions* A.1 and A.2 *hold for Model II. Then*

(1) *the equilibrium point* $(\delta_{\mathrm{ESS}};\,0)$ *is the ESS if there exists a minimum nonnegative value* δ_{ESS} *such that*

$$\begin{cases} x^*(\delta_{\mathrm{ESS}};0) \ge \theta_\delta, \\ H_\delta^+ - c_\delta(\delta_{\mathrm{ESS}}) \ge H_\delta^-, \\ H_\gamma^+ - C_\gamma(\theta_\gamma, \delta_{\mathrm{ESS}}) \le H_\gamma^-; \end{cases}$$

(2) *the equilibrium point* $(0;\,\gamma_{\mathrm{ESS}})$ *is the ESS if there exists a minimum nonnegative value* γ_{ESS} *such that*

$$\begin{cases} x^*(0;\,\gamma_{\mathrm{ESS}}) \le \theta_\gamma, \\ H_\gamma^+ - c_\gamma(\gamma_{\mathrm{ESS}}) \ge H_\gamma^-, \\ H_\delta^+ - C_\delta(\theta_\delta,\,\gamma_{\mathrm{ESS}}) \le H_\delta^-. \end{cases}$$

Example 3.15 For the data of Example 3.13, choose λ = 2, which yields no Nash equilibria in the game. From the first system of inequalities in Proposition 3.15 we find that $\delta_{\mathrm{ESS}} \approx 0.816$ implements the unit CBE. The second system of inequalities in Proposition 3.15 is infeasible, i.e., this ESS is unique. •

At the end of this paragraph dedicated to threshold goal functions, note that the choice of thresholds in the payoff functions of Principals and the payoffs themselves can be treated as *meta control*. Really, with a known relationship between the equilibrium of the Principals' game and these parameters, it is possible to analyze three-level models (meta level–Principals–agents), i.e., to choose the admissible values of the parameters in the Principals' game that lead to an equilibrium implementing the desired CBE in the agents' game. We give an illustrative example below.

Example 3.16 For the data of Example 3.13, choose $\lambda = 20$ and consider the following problem. It is required to choose values H_δ^+ and H_γ^+, for which the zero strategy profile becomes the NE in the Principals' game. By condition (3.61) it suffices to decrease H_δ^+ to 4.9×10^{-4}.

For the data of Example 3.13 and $\lambda = 20$, the next problem is to choose values H_δ^+ and H_γ^+ that implement the CBE $\theta_\gamma = 0.4$. In accordance with expression (3.63), it suffices to choose $H_\delta^+ \le 0.029$ and $H_\gamma^+ \ge 4$. •

In addition to the standard normal-form games, we will study their extensions, namely, hierarchical and reflexive games between two Principals. As a matter of fact, the forthcoming paragraphs merely demonstrate how the corresponding classes of the game-theoretic models of informational confrontation can be described and analyzed. Their systematic treatment is the subject of further research.

Hierarchical game of Principals

In mob control problems, the players (Principals) often make decisions sequentially. Here the essential factors are the awareness of each player at the time of decision-making and the admissible strategy sets of the players (for a classification and research of *hierarchical games*, we refer to the classical monograph [80]). A certain hierarchical game can be "superstructed" over each normal-form game [158, 160, 165]. Moreover, it is necessary to discriminate between two setups as follows:

(1) One of the Principals chooses his/her strategy and then the other does so, being aware of the opponent's choice. After that, an informational influence is applied on the agents. As a result, the distribution function of the agents' thresholds takes form (3.42) or (3.45). We will study this case below.
(2) One of the Principals chooses his/her strategy and applies his/her informational influence on the agents. After that, the other Principal chooses his/her strategy and applies his/her informational influence on the agents, being aware of the opponent's choice.

For Model I, both setups are equivalent as they yield the same distribution function (3.42) of the agents' thresholds. However, they differ within the framework of Model II.

In the games Γ_1 [71] (including the *Stackelberg games* [80, 153]), the admissible strategy sets of the Principals are the same as in the original normal-form game, and the Principal making the second move knows the choice of the opponent moving first. The corresponding situations can be interpreted as control and *countercontrol* (e.g., under a given value of α, choose β, or vice versa). If the original normal-form game can be easily analyzed with an explicit relationship between the equilibria and model parameters, then the corresponding game Γ_1 is often examined without major difficulties.

Consider several examples of hierarchical games for the first setup of Model I with the threshold payoff functions of Principals.

Example 3.17 For the data of Example 3.12 and $\theta = 1/3$, Principal 1 chooses the parameter α and then Principal 2 chooses the parameter β, being aware of the opponent's choice (the so-called game $\Gamma_1(\alpha, \beta)$). It follows from expressions (3.50) and (3.56) that

$$\Omega_{\alpha,\beta}(\theta) = \left\{(\alpha, \beta) \in [0,1]^2 \middle| \beta = \frac{\alpha(1-\theta)}{\alpha + \theta - 2\alpha\theta},\ 0 < \alpha,\ \beta \leq 1 - 1/e\right\}.$$

If Principal 1 chooses the strategy α^S, then the best response of Principal 2 has the form

$$\begin{aligned} \beta^S(\alpha^S) &= \arg\max_{\beta\in[0,1]} [H_\beta(x^*(\alpha^{S^*}(\alpha, \beta)) - c_\beta(\beta)] \\ &= \arg\max_{\beta\in[0,1]} \left[\begin{cases} 1 & \text{if } x^*(\alpha^S, \beta) \leq \theta, \\ 0 & \text{otherwise,} \end{cases} + \ln(1-\beta)\right] = \frac{2\alpha}{\alpha+1}. \end{aligned}$$

In other words, Principal 2 benefits from choosing the minimum value of β that implements the CBE θ given α^S. The goal function of Principal 1 can be written as $H_\alpha(x^*(\alpha^S, \beta^S(\alpha^S))) - c_\alpha(\alpha^S) = 1 - c_\alpha(\alpha^S)$, where $0 < \alpha \leq 1 - 1/e$. Therefore, the ε-optimal solution $(\alpha^{S^*}, \beta^{S^*})$ of the game $\Gamma_1(\alpha, \beta)$ is the pair of strategies $(\varepsilon, 2\varepsilon/(\varepsilon + 1))$ yielding the Principals' payoffs $1 + \ln(1 - \varepsilon)$ and $1 + \ln(1 - 2\varepsilon/(\varepsilon + 1))$, respectively. (Here ε represents an arbitrary small strictly positive value.) Note a couple of aspects as follows. First, this solution is close to the utilitarian solution, since both Principals choose almost zero strategies. Second, the Principal moving second incurs higher costs. •

Example 3.18 For the data of Example 3.17, Principal 2 chooses the parameter β and then Principal 1 chooses the parameter α, being aware of the opponent's choice (the so-called game $\Gamma_1(\beta, \alpha)$). It follows from expressions (3.50) and (3.56) that

$$\Omega_{\alpha,\beta}(\theta) = \left\{(\alpha, \beta) \in [0,1]^2 \middle| \alpha = \theta\beta/(1 - \beta - \theta + 2\beta\theta),\ 0 < \alpha,\ \beta \leq 1 - 1/e\right\}.$$

In this case, the ε-optimal solution of the game $\Gamma_1(\beta, \alpha)$ is the pair of strategies $(\varepsilon/(2 - \varepsilon), \varepsilon)$, yielding the Principals' payoffs $1 + \ln(1 - \varepsilon/(2 - \varepsilon))$ and $1 + \ln(1 - \varepsilon)$, respectively. Again, this solution is close to the utilitarian analog and the Principal moving second incurs higher costs. •

Based on Examples 3.17 and 3.18, we make the following hypothesis, which is well-known in theory of hierarchical games and their applications. The solutions of the games $\Gamma_1(\alpha, \beta)$ and $\Gamma_1(\beta, \alpha)$ belong to the domain of compromise (if nonempty), and the Principals compete for the first move: the Principal moving first generally compels the opponent "to agree" with a nonbeneficial equilibrium. This property appears in many control models of organizational systems (e.g., see [165]).

Now, consider the games Γ_2 in which the Principal moving first possesses a richer set of admissible strategies [71]: he/she chooses a relationship between his/her actions and the opponent's actions and then reports this relationship to the latter.

Using the ideology of Germeier's theorem [71], one can expect the following. If the domain of compromise is nonempty, the optimal strategy of Principal 1 (first choosing the parameter α, i.e., in the game $\Gamma_2(\alpha(\cdot),\ \beta)$) has the form

$$\alpha^{G^*}(\beta) = \begin{cases} \alpha^{S^*} & \text{if } \beta = \beta^{S^*}, \\ 1 & \text{otherwise.} \end{cases} \tag{3.64}$$

In a practical interpretation, the strategy (3.64) means that Principal 1 suggests the opponent to implement the solution $(\alpha^{S^*};\ \beta^{S^*})$ of the game $\Gamma_1(\alpha,\ \beta)$. If Principal 2 rejects the offer, Principal 1 threatens him/her with the choice of the worst-case response. The game $\Gamma_2(\alpha(\cdot),\ \beta)$ with strategy (3.64) leads to the same equilibrium payoffs of the Principals as the game $\Gamma_1(\alpha,\ \beta)$.

The game $\Gamma_2(\beta(\cdot),\ \alpha)$ as well as the hierarchical games for Model II are described by analogy.

Reflexive game of Principals

It is also possible to "superstruct" *reflexive games* [168] over a normal-form game in which players possess nontrivial mutual awareness about some essential parameters. Assume the distribution function $F(r,\ x)$ contains a parameter $r \in Y$ that describes uncertainty. Following the paper [168], denote by r_1 and r_2 the beliefs of Principals 1 and 2 about the parameter r, by r_{12} the beliefs of Principal 1 about the beliefs of Principal 2, and so on.

Example 3.19 For Model II, choose $F(r,x) = r + (1-r)\,x,\ r \in Y = [0,1]$, $H_\delta(x) = x$, $H_\gamma(x) = 1 - x$, $c_\delta(\delta) = \delta$, and $c_\gamma(\gamma) = \lambda\ \gamma$. The corresponding CBE is $x^*(\delta,\ \gamma) = (\delta + r)\ /\ (\delta + \gamma + r)$, and the Principals' goal functions take the form

$$f_\delta(\delta,\gamma) = (\delta + r)/(\delta + \gamma + r) - \delta, \tag{3.65}$$

$$f_\gamma(\delta,\gamma) = 1 - (\delta + r)/(\delta + \gamma + r) - \lambda^2\gamma. \tag{3.66}$$

If the parameter $r \in [0,1]$ is *common knowledge* [168] between the Principals, then expressions (3.65) and (3.66) yield the following parametric NE of the Principals' game:

$$\delta^* = \left(\frac{\lambda}{1+\lambda^2}\right)^2 - r, \tag{3.67}$$

$$\gamma^* = \frac{1}{(1+\lambda^2)^2}. \tag{3.68}$$

These strategies implement the CBE

$$x^*(\delta^*, \gamma^*) = \frac{\lambda^2}{1 + \lambda^2}. \tag{3.69}$$

Interestingly, the equilibrium strategy (3.68) of Principal 2 and the corresponding CBE (3.69) are independent of the parameter $r \in [0, 1]$ under common knowledge. The situation completely changes without the common knowledge about this parameter.

Let $r_1 = r_{12} = r_{121} = r_{1212} = \ldots$, i.e., Principal 1 possesses some (generally, incorrect) information r_1 about the uncertain parameter r, supposing that his/her beliefs are true and form common knowledge. Also, choose $r_2 = r_{21} = r_{212} = r_{2121} = \cdots = r$, i.e., Principal 2 is aware of the true value of r and considers it as common knowledge. In other words, Principal 2 does not know that the beliefs of Principal 1 possibly differ from the reality.

Using expressions (3.67) and (3.68), we calculate the *informational equilibrium* [165] of the Principals' game as

$$\delta_* = \left(\frac{\lambda}{1 + \lambda^2}\right)^2 - r_1, \ \gamma_* = \frac{1}{(1 + \lambda^2)^2},$$

which implements the CBE

$$x^*(\delta_*, \gamma_*) = \frac{\lambda^2 + (r - r_1)(1 + \lambda^2)^2}{1 + \lambda^2 + (r - r_1)(1 + \lambda^2)^2}. \tag{3.70}$$

Clearly, in the general case the CBE depends on the awareness of both Principals; under common knowledge $r_1 = r$, expression (3.70) takes form (3.69). Implementing *informational control* [165, 168] as *meta control* (e.g., affecting the beliefs of Principal 1 about the value of the uncertain parameter), we may accordingly change the CBE. •

Example 3.20 For the data of Example 3.19, let Principal 2 possess adequate awareness about the opponent's beliefs. That is, the former knows that the beliefs of Principal 1 may differ from the truth: $r_{21} = r_{212} = r_{2121} = \cdots = r_1$. Then in the informational equilibrium Principal 1 still prefers the strategy $\delta_* = \left(\frac{\lambda}{1 + \lambda^2}\right)^2 - r_1$, whereas Principal 2 chooses

$$\gamma_*(r_1, r) = \frac{1}{\lambda}\sqrt{\left(\frac{\lambda}{1 + \lambda^2}\right)^2 - r_1 + r} + r_1 - r - \frac{\lambda^2}{(1 + \lambda^2)^2},$$

which implements the CBE

$$x^*(\delta_*, \gamma_*(r_1, r)) = \lambda \frac{\lambda^2 + (r - r_1)(1 + \lambda^2)^2}{(1 + \lambda^2)^2 \sqrt{\left(\frac{\lambda}{1+\lambda^2}\right)^2 - r_1 + r}}.$$

Obviously, in the case of common knowledge ($r_1 = r$), we have $x^*(\delta_*, \gamma_*(r_1, r)) = x^*(\delta^*, \gamma^*)$.

Therefore, as shown by this example, equilibria in reflexive games also essentially depend on the *mutual awareness* of players, i.e., the beliefs about the opponents' awareness, the beliefs about their beliefs, and so on [168]. •

Besides, a nontrivial mutual awareness of Principals may cover not only the parameters of the distribution function of the agents' thresholds but also the parameters of their payoff and/or cost functions, etc.

Example 3.21 For the data of Example 3.19, let Principal 1 possess an inadequate awareness about the parameter λ of the opponent's cost function. In turn, Principal 2 knows the true value of this parameter, supposing the adequate awareness of Principal 1.

Choose $\lambda_1 = \lambda_{12} = \lambda_{121} = \lambda_{1212} = \ldots$, i.e., Principal 1 has some (generally, incorrect) information λ_1 about the uncertain parameter λ, considering his/her beliefs to be true and form common knowledge. Also choose $\lambda_2 = \lambda_{21} = \lambda_{212} = {}_{2121} = \cdots = \lambda$, i.e., Principal 2 knows the true value of the parameter λ, considering it as common knowledge. Using expressions (3.67) and (3.68), we obtain the CBE

$$x^* = \frac{\lambda_1^2}{\lambda_1^2 + \left(\frac{1+\lambda_1^2}{1+\lambda^2}\right)^2},$$

which is implemented in the corresponding informational equilibrium. In case of common knowledge ($\lambda_1 = \lambda$), it becomes the CBE (36). •

The main results of Sect. 3.3 are as follows. It has been demonstrated how the stochastic model of mob control [29] can be augmented by "superstructing" different game-theoretic models of interaction between control subjects that apply informational influences on a mob for their personal benefit. A relatively "simple" model of this controlled object (a mob) fits a rich arsenal of game theory, namely, normal-form games, hierarchical games, reflexive games and other games.

A promising direction of future investigations consists in identification and separation of typical distribution functions of the agents' thresholds (e.g., by analogy with the paper [17]). This would yield control templates and standard solutions for informational control problems as well as for models of informational confrontation.

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